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- The recent research monograph, 'Foundations of noncommutative function theory' by **Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov** in addition to containing numerous fundamental new results contains a panoramic survey of the field to date. arXiv:1212.6345.

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If $p \in \mathbb{P}_d$, then for each $n \in \mathbb{N}$ we can define a function $p^\wedge$ by plugging in matrices for the variables.

Thus, if p is the polynomial above, and $M = (M^1, M^2)$ is a pair of $n \times n$ matrices, $p^\wedge(M)$ is the $n \times n$ matrix defined by

$$p^\wedge(M) = 1 + 3M^1M^2 - 2M^2M^1 + 7M^1M^2M^1.$$

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$\mathcal{M}^d$ is the 'one dimensional nc-universe', $\mathcal{M}^d$ is the 'd-dimensional nc-universe' and if $p \in \mathbb{P}_d$, then

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   whenever both M and $S^{-1}MS$ are points in D.

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Definition. The **free topology on** $\mathcal{M}^d$ is the topology on $\mathcal{M}^d$ which has as a basis the basic open sets.

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5. ϕ is **analytic** if each for each $n \in \mathbb{N}$ each of the n^2 entries of $\phi(x)$ is an analytic function of the dn^2 entries of x.

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2 implies 3 is inspired by one of the many interesting ideas in J.W. Helton, I. Klep, and S. McCullough's *Proper Analytic Free Maps*, J. Funct. Anal. 260 (2011) 14761490.

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Example. If $\delta = (x^1x^2 - x^2x^1)^2 - 4$, then G_δ is a nonempty basic set and

$$\phi(x) = \sum_{k=1}^{\infty} \frac{\delta(x)^k}{k}$$

defines a free holomorphic function on G_δ that is neither rational nor locally representable by a power series at any point $x_0 \in G_\delta$.

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Say an NP problem with nodes $\lambda_1, \ldots, \lambda_m \in G_\delta$ and targets $z_1, \ldots, z_m \in \mathcal{M}^1$ is **solvable** on G_δ if there exists a ϕ satisfying 1, 2, and 3 above.

A Polynomial Solution Without Bounds

Lemma 1. If an NP problem with nodes $\lambda_1, \dots, \lambda_m \in G_\delta$ and targets $z_1, \dots, z_m \in \mathcal{M}^1$ is solvable, then it must be solvable (without bounds) by a free polynomial, i.e.,

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But, as $\mathcal{A}_\lambda$ is finite dimensional, $\mathcal{A}_\lambda^- = \mathcal{A}_\lambda$. Therefore, $z \in \mathcal{A}_\lambda$, i.e., there exists $\zeta \in \mathbb{P}_d$ such that $\zeta(\lambda) = z$. For this ζ, $\zeta(\lambda_i) = z_i$ for each i. $\square$

A Free Variety

A **free variety** is a set in $\mathcal{M}^d$ defined as the joint 0-set of a collection of free polynomials. Ideals that can be associated with these varieties are studied in 'Real Nullstellensatz and *-Ideals in *-Algebras' by J. Cimpric, J. W. Helton, S. McCullough, and C. Nelson (available on ArXiv)

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Lemma 2. If ϕ is any free holomorphic function on G_λ satisfying $\phi(\lambda) = \zeta(\lambda)$, then $\phi(x) = \zeta(x)$ for all $x \in V_\lambda \cap G_\delta$.

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If an NP problem with nodes $\lambda_1, \ldots, \lambda_m \in G_\delta$ and targets $z_1, \ldots, z_m \in \mathcal{M}^1$ is solvable, then necessarily there must be a free polynomial that interpolates the nodes to the targets, i.e.,

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Since a NP problem, if it is solvable, must be solvable by a polynomial, we may as well assume that the problem was presented to begin with via a polynomial, i.e., assume that the targets are given as the values at the nodes of a free polynomial.

The Free Nevanlinna-Pick Theorem

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$$\sup_{x \in V_\lambda \cap G_\delta} \|\zeta(x)\| \leq 1,$$

in the statement of the Free Nevanlinna Pick Theorem implies that ζ has a δ-**model** , i.e., there exist free vector-valued polynomials $p_1, \ldots, p_N \in \mathbb{C}^J \otimes \mathbb{P}_d$ such that

$$1 - \zeta(y)^*\zeta(x) = \sum_{k=1}^{N} p_k(y)^* \Big(1 - \delta(y)^*\delta(x)\Big) p_k(x), \qquad (x, y) \in E^{[2]}.$$

7. For each fixed $n \in \mathbb{N}$, apply a Lurking Isometry Argument to deduce the existence of an isometry

$$L_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} : \mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^I \otimes \ell^2) \to \mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^J \otimes \ell^2)$$

such that for each $x \in V_\lambda \cap G_\delta \cap \mathcal{M}_n^d$,

$$\zeta(x) = A_n + B_n \left(\delta(x) \otimes \mathrm{id}_{\ell^2}\right) \left(1 - D_n \left(\delta(x) \otimes \mathrm{id}_{\ell^2}\right) \right)^{-1} C_n.$$

8. Exploit the fact that ζ, the entries of δ, and the coefficients $p_1, \ldots, p_N$ in the model formula for ζ are free polynomials to show that the highly non-unique isometries L_n of the previous step can be modified so as to satisfy the additional property that for each $n \in \mathbb{N}$,

$$L_n = \mathrm{id}_n \otimes L_1.$$

9. Steps 7. and 8. imply the formula,

$$\zeta(x) = \mathrm{id}_n \otimes A_1 + (\mathrm{id}_n \otimes B_1)(\delta(x) \otimes \mathrm{id}_{\ell^2})\Big(1 - (\mathrm{id}_n \otimes D_1)(\delta(x) \otimes \mathrm{id}_{\ell^2})\Big)^{-1}(\mathrm{id}_n \otimes C_1),$$

valid for each $n \in \mathbb{N}$ and each $x \in V_\lambda \cap G_\delta \cap \mathcal{M}_n^d$.

10. Define ϕ on all of $G_\delta \cap \mathcal{M}_n^d$ by the formula,

$$\phi(x) = \mathrm{id}_n \otimes A_1 + (\mathrm{id}_n \otimes B_1)(\delta(x) \otimes \mathrm{id}_{\ell^2})\Big(1 - (\mathrm{id}_n \otimes D_1)(\delta(x) \otimes \mathrm{id}_{\ell^2})\Big)^{-1}(\mathrm{id}_n \otimes C_1),$$

11.
- $x \in G_\delta$ implies

$$\|(\mathrm{id}_n \otimes D_1)(\delta(x) \otimes \mathrm{id}_{\ell^2})\| < 1.$$

  Therefore, ϕ is well defined.
- The $\otimes$'s are in the formula are laid out in such a way that 'follow your nose algebra' guarantees that ϕ is an nc-function.
- Since $L_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ is an isometry,

$$\sup_{x \in G_\delta} \|\phi(x)\| \leq 1.$$

- In particular, the Fundamental Theorem implies that ϕ is a free holomorphic function.
- The formulas for ζ and ϕ agree when $x \in V_\lambda \cap G_\delta$. Therefore, since $\lambda_i \in V_\lambda \cap G_\delta$ for each i, $\phi(\lambda_i) = \zeta(\lambda_i)$ for each i.