On the Sarason conjecture for the Bergman space

joint with Sandra Pott and Maria Carmen Reguera

Let:
$\mathbb{D}$ be the unit disc in the complex plane,
A be the normalized area measure on $\mathbb{D}$.

The Bergman space L^2_a is the closed subspace of $L^2(\mathbb{D}, A)$ consisting of analytic functions in $\mathbb{D}$.

The orthogonal projection from $L^2(\mathbb{D}, A)$ onto L^2_a is given by

$$Pu(z) = \int_{\mathbb{D}} \frac{1}{(1-\bar{\zeta}z)^2} u(\zeta) dA(\zeta), \quad z \in \mathbb{D}. \tag{1}$$

The Toeplitz operator T_f with symbol $f \in L^2(\mathbb{D})$ is densely defined by

$$T_f u = P(fu)\,, \quad u \in L^2_a\,.$$

Then $T_f^* = T_{\bar{f}}$ and if f is analytic in $\mathbb{D}$ then

$$T_f u = fu\,.$$

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The problem

is to characterize the analytic symbols f, g for which $T_f T_g^*$ is bounded on L_a^2

Connection to the two weight problem

The inequality

$$|\langle T_f T_g^* u, v\rangle| \le C\|u\|_2\|v\|_2\,, \quad u, v \in L^2_a$$

means that

$$\left|\int_{\mathbb{D}} \overline{v} f P(\overline{g}u)\, dA\right| \le C\|u\|_2\|v\|_2\,, \quad u, v \in L^2_a$$

If f, g are analytic we can replace $u, v \in L^2_a$, by $u, v \in L^2(\mathbb{D})$ and the inequality remains equivalent to the previous one.

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If we let $x = u\overline{g}$, this can be rewritten as

$$\int_{\mathbb{D}} |fPx|^2 dA \leq C^2 \|x\|^2_{L^2(|g|^{-2}dA)} ,$$

i.e. P is a bounded operator from $L^2(|g|^{-2}dA)$ to $L^2(|f|^2dA)$.

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Of course the same holds for the Hardy space and the Riesz projection as well. More than that, in this case the weights $|g|^{-2}$ and $|f|^2$ can be quite general.

The boundedness of the Riesz projection acting between weighted L^2-spaces on the unit circle has attracted a lot of attention and became an important and difficult problem in harmonic analysis.

There are excellent contributions to the subject by Nazarov, Treil and Volberg, and quite recently by Lacey. Sawyer, and Uriarte-Tuero, and the same group in collaboration with Shen (2011+preprints).

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The Sarason condition

is inspired by the Bekollé-Bonami condition which characterizes the boundedness of P from $L^2(wdA)$ into itself.
The theorem of Bekollé-Bonami asserts that this is equivalent to

$$\sup_Q \frac{1}{A(Q)} \int_Q wdA \frac{1}{A(Q)} \int_Q w^{-1} dA < \infty\,,$$

whre Q denotes a Carleson square.

This can be rewritten in the conformal invariant form

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} w \circ \phi_z dA \int_{\mathbb{D}} w^{-1} \circ \phi_z dA < \infty$$

where $\phi_z(\zeta) = \frac{z-\zeta}{1-\bar{z}\zeta}$.

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The Sarason condition

or equivalently,

$$\sup_{z\in\mathbb{D}} B(w)(z)B(w^{-1})(z) < \infty\,,$$

where B denoteds the Berezin transform

$$B(u)(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{u(\zeta)}{|1-\bar{z}\zeta|^4}\, dA(\zeta)\,.$$

The Sarason condition

Saranson conjectured that $T_f T_g^*$ is bounded on L^2_a if and only if

$$\sup_{z\in\mathbb{D}} B(|f|^2)(z)B(|g|^2)(z) < \infty\,.$$

The corresponding conjecture for H^2 is similar, one simply replaces the Berezin transforms by Poisson integrals.

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The condition is necessary

Recall that $\phi_z(\zeta) = \frac{z-\zeta}{1-\bar{z}\zeta}$ and start with the identity

$$\frac{1}{(1-\bar{\lambda}\zeta)^2} = \phi_z'(\zeta)\overline{\phi_z'(\lambda)} + \frac{2\phi_z(\zeta)\overline{\phi_z(\lambda)} - 1}{(1-\bar{\lambda}\zeta)^2} .$$

The condition is necessary

If f, g are analytic this yields

$$2\langle T_f T_g^* u, v\rangle = \langle f\phi_z', v\rangle\langle u, g\phi_z'\rangle + 2\langle T_{\phi_z} T_f T_g^* T_{\phi_z}^* u, v\rangle\,,$$

and for the two weight problem

$$2\langle Px, y\rangle_{L^2(|f|^2dA)} = \langle \phi_z', y\rangle_{L^2(|f|^2dA)}\langle x, \phi_z'\rangle_{L^2(|g|^{-2}dA)} + 2\langle M_{\phi_z} P M_{\phi_z}^* x, y\rangle_{L^2(|f|^2dA)}\,,$$

where M_{ϕ_z} is the operator of multiplication by ϕ_z.

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where M_{ϕ_z} is the operator of multiplication by ϕ_z.

The condition is necessary

If we assume $T_f T_g^*$ bounded on L_a^2 then

$$|\langle f\phi_z', v\rangle\langle u, g\phi_z'\rangle| \leq C\|u\|_2\|v\|_2$$

and with the choice $u =, g\phi_z',\ v = f\phi_z'$ we obtain

$$B(|f|^2)(z)B(|g|^2)(z) \leq C^2 .$$

A similar argument yields the same conclusion if we assume that the Bergman projection is bounded from $L^2(|g|^{-2}dA)$ to $L^2(|f|^2dA)$.

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...but not sufficient

This was first done by Nazarov (1997) who constructed examples of:

- $f, g \in H^2$ such that $T_f T_g^*$ is not bounded on H^2, but the corresponding Sarason condition holds
- A radial function $f \in L^2(\mathbb{D})$ such that $B(|f|^2)$ is bounded but T_f is unbounded.

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Another example

Let

$$f(z) = 1 - |z|, \ z \in \mathbb{D}.$$

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Then a standard estimate gives

$$B(|f|^2)(z) \sim (1 - |z|)^2 \log \frac{2}{1 - |z|},$$

Given $g \in L^2(\mathbb{D})$, the Sarason condition

$$\sup_{z \in \mathbb{D}} B(|f|^2(z) B(|g|^2(z) < \infty$$

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Another example

By further standard estimates this is equivalent to

$$\int_Q |g|^2 dA \leq C \frac{1}{\log \frac{2}{\ell(Q)}},$$

for all Carleson squares Q, where $\ell(Q)$ denotes the perimeter of Q.

Another example

Proposition

If $f(z) = 1 - |z|,\ z \in \mathbb{D}$, and $g \in L^2_a$ then $T_f T_g^*$ is bounded if and only if $|g|^2 dA$ is a Carleson measure for the Dirichlet space.

Recall that this is equivalent to

$$\int_{\mathbb{D}} |g|^2 ||^2 dA \leq C \int_{\mathbb{D}} |u'|^2 dA$$

for all polynomials u with $u(0) = 0$.

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Another example

For f, g as above, the condition for the boundedness oh $T_f T_g^*$ is much more restrictive than the Sarason condition (Stegenga 1978), the problem was also extensively studied by Arcozzi, Rochberg, Sawyer, Wick.
In particular, this gives another example that the Sarason condition is not sufficient for the two weight problem.

Another example

Reason:

$$\langle T_f T_g^* u, v\rangle = \int_{\mathbb{D}} \overline{g} u(\zeta) \int_{\mathbb{D}} \frac{\overline{v(z)}}{(1-\overline{\zeta} z)^2}(1-|z|)dA(z)dA(\zeta)$$

When v runs in the unit ball of L_a^2 the complex conjugates of the functions given by the inner integral cover a ball (centered at the origin) in the Dirichlet space D, i.e. g satisfies

$$\int_{\mathbb{D}} \overline{gh} u dA \leq C\|u\|_2\|h\|_D .$$

Testing conditions

The argument used in the proof of the necessity of the Sarason condition works with approprate modifications for the Hardy space as well (in fact it originates from this situation). Together with Nazarov's example it shows that testing boundedness of $T_f T_g^*$ by

$$\frac{\sup_{z\in\mathbb{D}} |\langle T_f T_g^* g(\phi_z')^{1/2}, f(\phi_z')^{1/2}\rangle|}{\|g(\phi_z')^{1/2}\|\|f(\phi_z')^{1/2}\|}$$

is not sufficient.

A lot of effort has been directed towards finding other classes of simple functions that do the job. A very natural candiate is obtained by replacing $(\phi_z')^{1/2}$ by characteristic functions of arcs. By a real "tour de force" Lacey, Sawyer, Shen, Uriarte-Tuero came very close to this result (Last preprint!)

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Main resultt

Theorem

For $f, g \in L^2_a$ the following are equivalent:
(i) $T_f T_g^*$ is bounded,
(ii) P is bounded from $L^2(|g|^{-2} dA)$ to $L^2(|f|^2 dA)$,
(iii) There exists $C > 0$ such that for all Carleson squares Q we have

$$\int_Q |f(z)|^2 \left(\int_Q \frac{|g(\zeta)|^2}{|1 - \bar{\zeta} z|^2} dA(\zeta) \right)^2 dA(z) \le C \int_Q |g|^2 dA,$$

and

$$\int_Q |g(z)|^2 \left(\int_Q \frac{|f(\zeta)|^2}{|1 - \bar{\zeta} z|^2} dA(\zeta) \right)^2 dA(z) \le C \int_Q |f|^2 dA.$$

The testing conditions in the theorem come from the analysis of a different operator, *the maximal Bergman projection*

$$P^{+}u(z) = \int_{\mathbb{D}} \frac{u(\zeta)}{|1-\bar{\zeta}z|^2} dA(\zeta)\,.$$

We want to solve the two weight problem for P^{+}, that is, to characterize its boundedness from $L^2(|g|^{-2}dA)$ to $L^2(|f|^2 dA)$.

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We want to solve the two weight problem for P^{+}, that is, to characterize its boundedness from $L^2(|g|^{-2}dA)$ to $L^2(|f|^2dA)$.

The tool is a dyadic model

$$P^{\beta}u = \sum_{Q \in \mathcal{D}_{\beta}} \chi_Q \frac{1}{A(Q)} \int_Q u dA\,,$$

where $\mathcal{D}_{\beta}$ is a set of Carleson squares based on a dyadic grid of arcs on the circle which is essentially obtained from a fixed one by means of a rotation with angle β. Moreover, χ_Q is the characteristic function of Q.

This type of operator originates from the work of Sawyer. When the Q's are dyadic cubes in $\mathbb{R}^n$, Lacey, Sawyer and Uriarte-Tuero (2011) solved the two weight problem for "all" such operators by means of testing conditions.

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The following result is an adaptation of their theorem to the disc and to the special operator P^β.

Proposition

P^β is bounded from $L^2(|g|^{-2}dA)$ to $L^2(|f|^2 dA)$ if and only if there exists $C > 0$ such that for all $Q \in \mathcal{D}_\beta$ we have

$$\int_Q |f|^2 \left(\sum_{\substack{S \in \mathcal{D}_\beta \\ S \subset Q}} \chi_S \frac{1}{A(S)} \int_S |g|^2 dA \right)^2 dA \leq C \int_Q |g|^2 dA,$$

and

$$\int_Q |g|^2 \left(\sum_{\substack{S \in \mathcal{D}_\beta \\ S \subset Q}} \chi_S \frac{1}{A(S)} \int_S |f|^2 dA \right)^2 dA \leq C \int_Q |f|^2 dA.$$

The continuous case

follows by the following inequality:

There exist β_0, β_1 and $C_0 C_1 > 0$ such that for nonnegative locally integrable functions u on $\mathbb{D}$ we have

$$C_0 P^{\beta_j} u \leq P^+ u \leq C_1 (P^{\beta_0} u + P^{\beta_1} u) , \quad j = 0, 1 .$$

This way of passing from dyadic to continuous has a beautiful history and holds in very general contexts (Petermichl, Hytönen, Lerner)

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The continuous case

Corollary

P^+ is bounded from $L^2(|g|^{-2}dA)$ to $L^2(|f|^2dA)$ if and only if there exists $C > 0$ such that for all Carleson squares Q

$$\int_Q |f(z)|^2 \left(\int_Q \frac{|g(\zeta)|^2}{|1-\bar{\zeta}z|^2} dA(\zeta) \right)^2 dA(z) \leq C \int_Q |g|^2 dA\,,$$

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Theorem

Let $f, g \in L^2_a$. Then P^+ is bounded from $L^2(|g|^{-2}dA)$ to $L^2(|f|^2 dA)$ if and only if $T_f T_g^*$ is bounded.

And now...?

Theorem

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Note that if $T_f T_g^*$ is bounded then

$$P_{f,g}u(z) = |f(z)| \int_{\mathbb{D}} \frac{|g(\zeta)|u(\zeta)}{(1-\bar{\zeta}z)^2} dA(\zeta)$$

is bounded on $L^2(\mathbb{D})$.

We need to prove that

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This cannot be done in the usual way , because we cannot control the oscillation of $|f|, |g|$ on pseudo-hyperbolic discs.

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The identities

$$\frac{1}{|1-\overline{\zeta}z|^2} = -\frac{\overline{\zeta}z}{(1-\overline{\zeta}z)^2} + \frac{1-|z\zeta|^2}{(1-\overline{\zeta}z)|1-\overline{\zeta}z|^2}$$
$$= -\mathrm{Re}\frac{\overline{\zeta}z}{(1-\overline{\zeta}z)^2} + \frac{1-|z\zeta|^2}{2|1-\overline{\zeta}z|^2} + \frac{(1-|z\zeta|^2)^2}{2|1-\overline{\zeta}z|^4}\,.$$

allow us to o write

$$\Delta(P^+_{f,g}u(z))^2 = G_1(z) + B(z)$$

$$\frac{1}{z}\overline{\partial}(P^+_{f,g}u(z))^2 = G_2(z) - \frac{B(z)}{1-|z|^2}$$

where the G's are good terms that can be estimated with help of $P_{f,g}u$, while the bad term B cannot.

However, by Stokes's formula

$$\int_{\mathbb{D}} (\Delta(P^{+}_{f,g}u(z))^2(1-|z|^2)^2 + \frac{1}{z}\overline{\partial}(P^{+}_{f,g}u(z))^2(1-|z|^2))dA(z)$$

$$= \int_{\mathbb{D}} (G_1(z)(1-|z|^2)^2 + G_2(z)(1-|z|^2))dA(z)$$

$$= \int_{\mathbb{D}} (P^{+}_{f,g}u(z))^2(4|z|^2-3)dA(z)$$

and this suffices to prove the theorem.