Two Variable Transfer Function Realizations and Agler Kernels

Kelly Bickel & Greg Knese
Washington University in St. Louis
St Louis, MO

Hilbert Function Spaces
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One-Variable Transfer Function Realizations

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Let $\phi \in \mathcal{S}(\mathbb{D})$. Then there is a Hilbert space $\mathcal{M}$ and a coisometry $V : \mathbb{C} \oplus \mathcal{M} \to \mathbb{C} \oplus \mathcal{M}$ so that if

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ \mathcal{M} \end{pmatrix}$$

then

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Importance

- Connects systems theory, operator theory, and function theory
- Basis for approaches to function theoretic problems on $\mathbb{D}$

Notation

1. A function $K : \Omega \times \Omega \to \mathbb{C}$ is a **positive kernel,** i.e. $\forall \, \{z^1, \dots, z^m\} \subset \Omega$,

$$\left(K(z^i, z^j)\right)_{i,j=1}^{m} \geq 0.$$

2. A Hilbert space of functions $\mathcal{H}(K)$ is a **reproducing kernel Hilbert space with kernel** K on Ω. The kernel functions $K_w(z) := K(z, w)$ are in $\mathcal{H}(K)$ and

$$\langle f, K_w \rangle_{\mathcal{H}(K)} = f(w) \qquad \forall \, f \in \mathcal{H}(K) \text{ and } w \in \Omega.$$

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Example. If $\phi \in \mathcal{S}(\mathbb{D})$, then

$$\mathcal{H}\left(\frac{1 - \phi(z)\overline{\phi(w)}}{1 - z\bar{w}}\right)$$

is the **de Branges-Rovnyak space associated to** ϕ.

de Branges-Rovnyak Spaces

Let $\mathcal{H}_\phi$ be the de Branges-Rovnyak space and K_ϕ the reproducing kernel. Then

$$\mathcal{H}_\phi := \mathcal{H}(K_\phi) = \mathcal{H}\left(\frac{1-\phi(z)\overline{\phi(w)}}{1-z\bar{w}}\right).$$

The functions $K_w(z) := K_\phi(z,w)$ are in $\mathcal{H}_\phi$ and

$$1-\phi(z)\overline{\phi(w)} = (1-z\bar{w})K_\phi(z,w) = (1-z\bar{w})\langle K_w, K_z\rangle_{\mathcal{H}_\phi} \quad \forall\ z,w \in \mathbb{D}.$$

Constructing Realizations

de Branges-Rovnyak Spaces

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Construction

Define

$$V : \begin{pmatrix} 1 \\ \bar{w}K_w \end{pmatrix} \to \begin{pmatrix} \overline{\phi(w)} \\ K_w \end{pmatrix} \qquad \forall\, w \in \mathbb{D}.$$

V is isometric and extends to an isometry on $\mathbb{C} \oplus \mathcal{H}_\phi$. Then ϕ has a transfer function realization with V^* and $\mathcal{H}_\phi$ is the state space.

One-Variable Extensions

Question: When does the transfer function realization extend to $\mathbb{T}$?

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Theorem 1 (Sz.-Nagy & Foias, 1970)

Let $\phi \in \mathcal{S}(\mathbb{D})$ and let X be an open set of $\mathbb{T}$. Then TFAE:

(i) Every function in $\mathcal{H}_\phi$ can be analytically continued across X.

(ii) ϕ can be analytically continued across X with unit modulus on X.

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(i) Every function in $\mathcal{H}_\phi$ can be analytically continued across X.

(ii) ϕ can be analytically continued across X with unit modulus on X.

Note: Replace (ii) with "ϕ extends continuously to X and with unit modulus on X." Also, $K_\phi(z, w)$ extends continuously to $X \times X$.

Conclusion: If ϕ extends continuously to X with unit modulus, then

$$\phi(z) = A + zBK_z \qquad \forall\ z \in \mathbb{D} \cup X.$$

Main Question:

How does this extension result generalize to two-variables?

Two-Variable Realizations

Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ be a Hilbert space and define $E_z : \mathcal{M} \to \mathcal{M}$ by

$$E_z = \begin{pmatrix} z_1 I_{\mathcal{M}_1} & 0 \\ 0 & z_2 I_{\mathcal{M}_2} \end{pmatrix} \qquad \forall\ z \in \mathbb{D}^2.$$

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Transfer Function Realizations (Agler '90)

$\phi \in \mathcal{S}(\mathbb{D}^2)$ iff there is a Hilbert space $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and a coisometry $V : \mathbb{C} \oplus \mathcal{M} \to \mathbb{C} \oplus \mathcal{M}$ such that if

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Uses

- Technique for generalizing function theoretic results from $\mathbb{D}$ to $\mathbb{D}^2$.
- Provides method to construct functions on $\mathbb{D}^2$ with desired properties.

Structure of Realizations

In one-variable:

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Two-variable Generalization (Agler, '90)

Let $\phi \in \mathcal{S}(\mathbb{D}^2)$. Then there are positive kernels K_1 and K_2 such that

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1\bar{w}_1)K_1(z, w) + (1 - z_2\bar{w}_2)K_2(z, w).$$

(K_1, K_2) are called **Agler kernels** of ϕ.

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Construction

Define $K_{j,w}(z) := K_j(z,w)$ and

$$V : \begin{pmatrix} 1 \\ \bar{w}_1 K_{1,w} \\ \bar{w}_2 K_{2,w} \end{pmatrix} \to \begin{pmatrix} \overline{\phi(w)} \\ K_{1,w} \\ K_{2,w} \end{pmatrix} \qquad \forall\ w \in \mathbb{D}^2.$$

V extends isometrically to $\mathbb{C} \oplus \mathcal{H}(K_1) \oplus \mathcal{H}(K_2)$ $(\oplus \mathcal{H})$. Then V^* is the desired coisometry with state space $\mathcal{H}(K_1) \oplus \mathcal{H}(K_2) \oplus \mathcal{H}$.

Interpreting Agler Kernel

Setup. If (K_1, K_2) are Agler kernels of ϕ, then

$$\frac{1-\phi(z)\overline{\phi(w)}}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)} = \frac{K_2(z,w)}{1-z_1\bar{w}_1} + \frac{K_1(z,w)}{1-z_2\bar{w}_2}.$$

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Characterization. As each term is a positive kernel:

$$\mathcal{H}_\phi := \mathcal{H}\left(\frac{1-\phi(z)\overline{\phi(w)}}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)}\right) \quad H_1 := \mathcal{H}\left(\frac{K_2(z,w)}{1-z_1\bar{w}_1}\right) \quad H_2 := \mathcal{H}\left(\frac{K_1(z,w)}{1-z_2\bar{w}_2}\right).$$

$\mathcal{H}_\phi$ is the two-variable **de Branges-Rovnyak space associated to** ϕ and

(1) The kernels of the H_j add to the kernel of $\mathcal{H}_\phi$.

(2) $z_j H_j \subseteq H_j$ and multiplication by z_j is a contraction.

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Idea: If H_1 and H_2 satisfy (1) and (2), then the numerators of their reproducing kernels are Agler kernels of ϕ.

Construction of Agler Kernels I

Ball, Sadosky, & Vinnikov's Work

$$\mathcal{K}_\phi = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f \in H^2, g \in L^2 \ominus H^2, f - \phi g \in (1 - |\phi|^2)^{1/2} L^2 \right\} \subseteq \mathrm{Im} \begin{bmatrix} I & \phi \\ \bar{\phi} & I \end{bmatrix}^{1/2},$$

where L^2 is the space of measurable, square-integrable functions on $\mathbb{T}^2$ and H^2 is the Hardy space on $\mathbb{D}^2$. Then

$$\mathcal{K}_\phi = S_1^{max} \oplus S_2^{min}$$

where S_1^{max} is the largest z_1-invariant subspace of $\mathcal{K}_\phi$ and $S_2^{min} := \mathcal{K}_\phi \ominus S_1^{max}$.

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Lemma 1 (Ball-Sadosky-Vinnikov, '05)

$$z_1 S_1^{max} \subseteq S_1^{max} \quad z_2 S_2^{min} \subseteq S_2^{min}$$

Multiplication by z_1 and z_2 are isometries on S_1^{max} and S_2^{min} respectively.

Construction of Agler Kernels II

There is a surjective, partial isometry $T : \mathcal{K}_\phi \to \mathcal{H}_\phi$.

$$H_1 := T(S_1^{max}) \quad \text{and} \quad H_2 := T(S_2^{min}).$$

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If H_1 and H_2 are endowed with certain inner products, then

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and the multiplication operators are contractions. Then, there are kernels K_1^{max} and K_2^{min} such that

$$H_1 = \mathcal{H}\left(\frac{K_1^{max}(z,w)}{1 - z_1\bar{w}_1}\right) \quad \text{and} \quad H_2 = \mathcal{H}\left(\frac{K_2^{min}(z,w)}{1 - z_2\bar{w}_2}\right).$$

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Also, the kernels of H_1 and H_2 add to the kernel of $\mathcal{H}_\phi$.

Conclusion: (K_1^{max}, K_2^{min}) are concrete Agler kernels of ϕ with specific structure.

Extensions of Kernels and Realizations

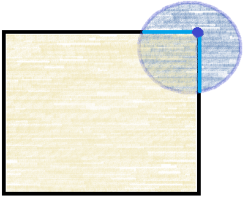
Theorem 1: Agler Kernel Extensions

Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ and let X be an open set of $\mathbb{T}^2$. Then TFAE:

(i) The functions in $\mathcal{H}(K_1^{max})$ and $\mathcal{H}(K_2^{min})$ extend analytically to a domain Ω containing X.

(ii) ϕ extends continuously to X and has unit norm there.

Given either condition, the kernels K_1^{max} and K_2^{min} extend continuously to $X \times X$.



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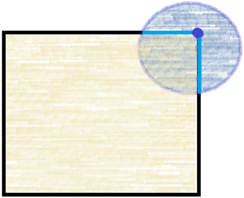

Conclusion: If ϕ extends continuously to $X \subseteq \mathbb{T}^2$ with unit modulus, then its transfer function realization associated to $\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})$ extends to X.

Thank you!