Generalized Hardy spaces in $\mathbb{R}^d$, products, paraproducts and div-curl lemma.

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Products of functions in $\mathcal{H}^1(\mathbb{R}^d)$ and BMO.

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Theorem [B., Iwaniec, Jones, Zinsmeister 2007] [B., Grellier, Ky 2012]. *The product $b \times h$, with $b \in \mathrm{BMO}$ and $h \in \mathcal{H}^1$ can be given a meaning as a distribution, and there are two bilinear operators, S and T, such that $bh = S(b,h) + T(b,h)$, with S and T continuous operators,*

$$S : \mathrm{BMO} \times \mathcal{H}^1 \mapsto \mathrm{L}^1(\mathbb{R}^{\mathrm{d}}), \qquad T : \mathrm{BMO} \times \mathcal{H}^1 \mapsto \mathcal{H}^{\log}(\mathbb{R}^{\mathrm{d}}).$$

What is $\mathcal{H}^{\log}$?

$$\mathcal{H}^{\log} := \{f \in \mathcal{S}'(\mathbb{R}^d)\ ;\ \int_{\mathbb{R}^d} \frac{\mathcal{M}f(x)}{\log(e+|x|)+\log(e+\mathcal{M}f(x))} dx < \infty.\}.$$

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Particular case of Hardy-Orlicz spaces of Musielak type introduced by Luong Dang Ky:
For $\Phi(x,t)$ having adequate properties, L^Φ is the space of functions such that

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For $\mathcal{H}^{\log}$, we have $\Phi(x,t) = \frac{t}{\log(e+|x|)+\log(e+t)}$.

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- of lower type $p < 1$: $\Phi(x,st) \le Cs^p\Phi(x,t)$ for $s < 1$.
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Luxembourg norm:

$$\|f\|_{L^\Phi} := \inf\left\{\lambda > 0 : \int \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Theorem [Ky]. *The space $\mathcal{H}^\Phi$ has an atomic decomposition. Its dual is the space BMO^Φ. When $p > \frac{n}{n+1}$, b is in BMO^Φ if and only if*

$$\sup_Q \frac{1}{\|\chi_Q\|_{L^\Phi}} \int_Q |b - b_Q| \, dx < \infty.$$

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$\mathrm{BMO}^{\log}$ already considered by Nakai and Yabuta:

For $b \in \mathrm{BMO}$, multiplication by b maps $\mathrm{BMO}^{\log} \cap \mathrm{L}^\infty$ into BMO.

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Equivalent characterizations as for Hardy spaces (Dachun Yang et al.)

Why the logarithm?

Use of the norm $\|b\|_{\mathrm{BMO}^+} = \|b\|_{\mathrm{BMO}} + \int_{(0,1)^d} |b| dx$ (we are not dealing with equivalent classes modulo constants!).

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Hölder type Inequality for this kind of exponential integrability condition and L^1 condition

Paraproducts, or Dobyinsky-Meyer renormalization

S and T may be given in a wavelet decomposition. For P_j associated to a MRA and $Q_j = P_{j+1} - P_j$, then

$$hb = \sum_{j\in\mathbb{Z}} (Q_j h)(Q_j b) + \sum_{j\in\mathbb{Z}} (P_j h)(Q_j b) + \sum_{j\in\mathbb{Z}} (Q_j h)(P_j b).$$

The first term is put in S, the last one in T. The middle term is in $\mathcal{H}^1$.

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The last paraproduct maps $\mathcal{H}^1$ into $\mathcal{H}^{\log}$.

An equivalence for commutators

Theorem [Ky]. *Let $b \in \mathrm{BMO}$ and $h \in \mathcal{H}^1$. Then it is equivalent that*

- *All commutators $[b, K]h$, for K a Calderón-Zygmund operator, are in L^1.*
- *Commutators $[b, R_j]h$, for R_j Riesz transforms, $j = 1, \cdots d$ are in L^1.*
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Pérez has defined via its atomic decomposition a subspace $\mathcal{H}_b$ such that $h \in \mathcal{H}_b$ satisfies these conditions.

Holomorphic Generalized Hardy spaces

We restrict now to Dimension 1 and define $\mathcal{H}^{\Phi}_{\mathrm{hol}}$ as the space of functions f that are holomorphic in the upper half-plane and such that

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Again, classical equivalence of definitions generalize. Recall that the dual of $\mathcal{H}^1_{\mathrm{hol}}$ is the space $\mathbb{BMOA}$.

Factorization Theorem [B., Ky]. *The pointwise product maps* $\mathbb{BMOA} \times \mathcal{H}^1_{\mathrm{hol}}$ *onto* $\mathcal{H}^{\log}_{\mathrm{hol}}$.

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Due to [B., Iwaniec, Jones, Zinsmeister] for the disc instead of the upper half-plane.

Hankel operators

The Hankel operator $\mathfrak{h}_b$ of symbol b is defined by

$$\langle \mathfrak{h}_b(f), g \rangle = \langle b, fg \rangle.$$

Corollary. *$\mathfrak{h}_b$ maps $\mathcal{H}^1$ into itself if and only if $b \in BMO^{\log}$.*

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Well known for the unit disc [Janson, Tolokonnikov], with generalizations to the unit ball [B., Grellier, Sehba] and to the polydisc [Pott, Sehba, $\cdots$].