Generalized Carleson-Newman inner functions

The pseudohyperbolic distance

$$\rho(z,w) = \left| \frac{z-w}{1-\bar{z}w} \right|$$

in the unit disc $\mathbb{D}$

Interpolating Blaschke products

$$B = \prod_{\lambda\in\Lambda} -\frac{|\lambda|}{\lambda} \frac{z-\lambda}{1-\bar{\lambda} z}$$

$$H^\infty|\Lambda = \ell^\infty(\Lambda)$$

$$\Updownarrow$$

$$\inf_{\lambda\in\Lambda} |B'(\lambda)|(1-|\lambda|^2) > 0$$

$$\Updownarrow$$

$$\inf_{\lambda_1,\lambda_2\in\Lambda,\ \lambda_1\neq\lambda_2} \rho(\lambda_1,\lambda_2) > 0$$

(pseudohyperbolic separation)

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$$\sup_{z\in\mathbb{D}} \sum_{\lambda\in\Lambda} \frac{(1-|z|^2)(1-|\lambda|^2)}{|1-z\bar{\lambda}|^2} < \infty$$

(Carleson embedding property)

12-02 Vasyunin (1978):

B interpolating $\Leftrightarrow$

$$|B(\lambda)| \geq C\rho(\lambda, Z(B)),$$

where $Z(B)$ = the zero set of B in $\mathbb{D}$.

Gorkin-Mortini (2010):

B is a Carleson-Newman Blaschke product (= the product of N interpolating products) $\Leftrightarrow$

$$|B(\lambda)| \geq C\rho^N(\lambda, Z(B)).$$

In 2008 Gorkin-Mortini-Nikolski introduced a class of generalized Carleson-Newman inner functions

$$\forall \varepsilon > 0 \ \exists \delta > 0: \ |\theta(\lambda)| < \delta \Rightarrow \rho(\lambda, Z(\theta)) < \varepsilon$$

$\Leftrightarrow$

$$|\theta(\lambda)| \geq \varphi(\rho(\lambda, Z(\theta)))$$

for some function $\varphi > 0$.

3 12-05 Equivalently,

$$\sup_{\rho(z, Z(\theta))>\varepsilon} \sum_{\lambda \in Z(\theta)} \frac{(1-|z|^2)(1-|\lambda|^2)}{|1-z\bar{\lambda}|^2} < \infty \quad \forall \varepsilon > 0$$

(weak embedding property)

<u>Motivation</u>:

$\Leftrightarrow$ $H^\infty/\theta H^\infty$ satisfies the norm controlled inversion property

$$\mathcal{H}(\varepsilon, \theta) = \sup\{\|1/f\|_{H^\infty/\theta H^\infty} : \varepsilon \le \|f|Z(\theta)\|_\infty \le \|f\|_{H^\infty/\theta H^\infty} \le 1\} < \infty \quad \forall \varepsilon > 0$$

$\Leftrightarrow$ $\forall f \in H^\infty$, $f(M_\theta)$ is invertible $\Leftrightarrow$ $\{f(\lambda)\}_{\lambda \in Z(\theta)}$ is bounded away from zero

(M_θ acts on $K_\theta = H^2 \ominus \theta H^2$,

$$M_\theta f = P_{K_\theta} z f$$

the model operator)

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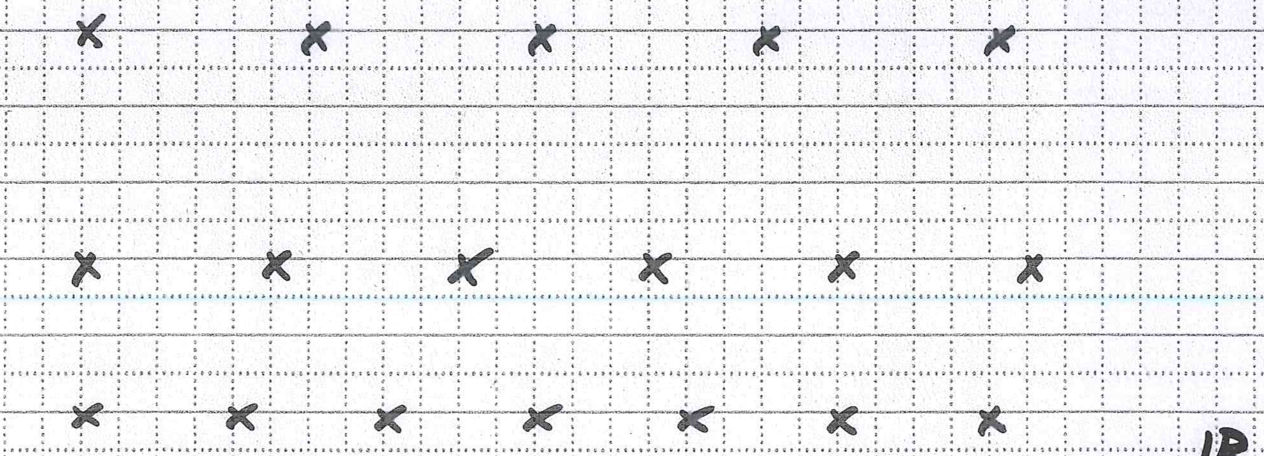
$\Longleftrightarrow Z_{\mathfrak{M}(H^\infty)}(\Theta) = \operatorname{clos}_{\mathfrak{M}(H^\infty)} Z(\Theta)$

$\Longleftrightarrow$ the quotient algebra $H^\infty / \Theta H^\infty$ has no corona

- $\{$GCN Blaschke products$\} \supsetneqq \{$CN Blaschke products$\}$

Treil's example

$$\Lambda = \{kn^2 + in^3\}_{k \in \mathbb{Z},\, n \geq 1} \subset \mathbb{C}_+$$



5 12-11

- B a Blaschke product w/ zero set Λ,

  $|B(z)| \gtrsim A\rho^N(z,\Lambda)$, $z\in\mathbb{D}$

  $\Rightarrow$ B = the product of N interpolating Blaschke products

Proof $\quad \mathcal{D}(z,\delta)=\{w\in\mathbb{D}:\rho(z,w)<\delta\}$

(a) $\operatorname{card}(\Lambda\cap\mathcal{D}(z,1/2))\le M(N,A)$

(b) for some $\delta=\delta(N,A)>0$,

$\operatorname{card}(\Lambda\cap\mathcal{D}(z,\delta))\le N$, $z\in\mathbb{D}$

$|B|\le\delta^{N+1}\Rightarrow \text{dist}\lesssim\delta^{1+\frac{1}{N}}$

$\Rightarrow \operatorname{card}\gtrsim\delta^{-1/N}$

$\Rightarrow$ easy separation

- $GCN\setminus CN$ is not stable under Frostman shifts (Mortini - Gorkin - Suarez):

  U = GCN inner, $|U(z)|\ge\Psi(\rho(z,Z(U)))$

  $0<|\gamma|<\sup_{t<1}\Psi(t)\Rightarrow\varphi_\gamma(U)\in CN$ Blaschke product

$\varphi_w(z)=(z-w)/(1-z\bar{w})$

6 | 12.15

Constructing Blaschke products in $GCN \setminus CN$

$\mathbb{D} = \bigcup Q_{jk}$,

$Q_{jk} = \{1-2^{-j+1} \le |z| < 1-2^{-j},$

$2\pi(k-1)2^{-j} \le \arg z \le 2\pi k 2^{-j}\}$,

$j \ge 1, \ 1 \le k \le 2^j$

Fix $N \ge 1$, choose Σ_N:

$\operatorname{card}(\Sigma_N \cap Q_{jk}) = (N-j)^+$

$\sup\limits_{w \in Q_{jk}} \rho(w, \Sigma_N) \asymp (N-j)^{-1/2} \qquad j<N$

$\rho(w, \Sigma_N \setminus \{w\}) \asymp (N-j)^{-1/2}$,

$j<N, \ w \in Q_{jk} \cap \Sigma_N$

$$B_N(w) = \prod_{z \in \Sigma_N} \frac{w-z}{1-w\bar{z}}$$

Then $|B_N(w)| > \Psi(\rho(w, \Sigma_N))$,

$\Psi(x) = c_1 x \exp\left(-\frac{c}{x^4}\right)$.

$\prod B_N(\varphi_{w_N}) \in GCN \setminus CN$

if $|w_N| \to 1$ sufficiently rapidly

• The indicator function

$$\eta_B(\varepsilon) = \sup\{\eta > 0 : |B(z)| < \eta \Rightarrow \rho(z, Z(B)) < \varepsilon\}$$

may vanish arbitrarily rapidly:

$\forall\ \psi\uparrow : (0,1) \to (0,1)$

$\exists\ B \in (GCN)$:

$$\eta_B = o(\psi) \text{ at } 0.$$

• Correspondingly,

$\mathcal{H}(\varepsilon, B)$ may grow arbitrarily fast as a function of ε

$$\left(= \sup\{\|1/f\|_{H^\infty/BH^\infty} : \varepsilon \le \|f\,|\,Z(B)\|_\infty \le \|f\|_{H^\infty/BH^\infty} \le 1\}\right)$$

• There exist B, B^* with the zero sets (z_k), (z_k^*) such that

$$\sup_k \rho(z_k, z_k^*) < 1$$

$$B \in (GCN), \quad B^* \notin (GCN)$$

17.21

Question : given B (or Θ) does there exist B_1 such that BB_1 (or ΘB_1) $\in (GCN)$?

- $U \in H^\infty$ satisfies the weighted area condition (A) if $\forall \varepsilon > 0$,

$$\int_{\Delta(U,\varepsilon)} \frac{dm_2(z)}{1-|z|} = \infty$$

where $\Delta(U,\varepsilon) = \{z : |U(z)| < \varepsilon\}$

- $U \in (A) \Rightarrow \nexists$ Blaschke product B such that $BU \in (GCN)$.

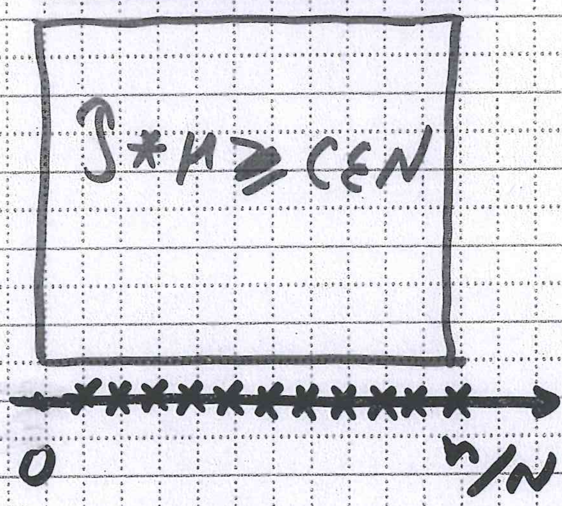
Proof $BU \in (GCN) \Rightarrow \exists \varepsilon > 0$:

$$\{\lambda : |BU(\lambda)| < \varepsilon\} \subset \bigcup_{\lambda \in Z(BU)} D(\lambda, 1/2) =: D(Z(BU), 1/2)$$

$$\int_{D(Z(BU),1/2)} \frac{dm_2(z)}{1-|z|} \le C \sum_{\lambda \in Z(BU)} 1-|\lambda| < \infty.$$

Theorem $\exists$ a singular inner function $S \in (A)$. $\nexists$ Blaschke product B such that $BS \in (GCN)$.

<u>Construction</u>



$\varepsilon > 0, \quad 0 < n \leq N$

$$\mu = \mu_{\varepsilon, n, N} = \varepsilon \sum_{1 \leq k \leq n} \delta_{e^{2\pi i k/N}}$$

$$\|\mu\| = n\varepsilon$$

$$\int_{S*\mu \geq c\varepsilon N} \frac{dm_2(z)}{1-|z|} \geq \frac{c\, n \log n}{N}$$

Choose: $\dfrac{n_k \log n_k}{N_k} \to \infty, \quad \varepsilon_k N_k \to \infty$

$$\sum \varepsilon_k n_k < \infty$$

$$\mu = \sum_k \mu_{\varepsilon_k, n_k, N_k}$$

- Such μ has large support

<u>def</u> $\displaystyle \operatorname{Ent} E = \sum_{\mathbb{T} \setminus E = \sqcup I} |I| \log \frac{1}{|I|}$

10

12-29

Theorem (Nicolau – Thomas)

$E \subset \mathbb{T}$, $|E| = 0$, E closed

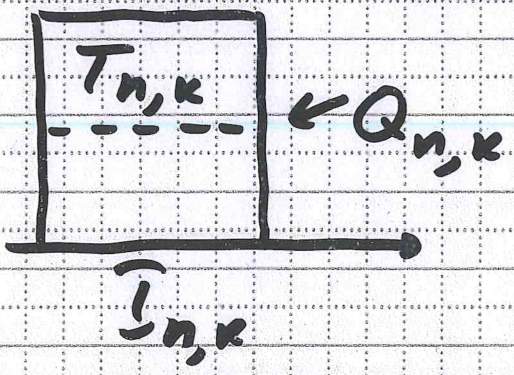
(a) $\operatorname{Ent} E < \infty \Rightarrow \forall \mu$ supported by E, $\forall \varepsilon > 0$,

$$\int_{\mathcal{P} * \mu > \varepsilon} \frac{dm_2(z)}{1 - |z|} < \infty$$

(b) $\operatorname{Ent} E = \infty \Rightarrow \exists \mu$ supported by E : $\forall \varepsilon > 0$,

$$\int_{\mathcal{P} * \mu > \varepsilon} \frac{dm_2(z)}{1 - |z|} = \infty$$

- $$\mathbb{T} = \bigcup_{0 \le k < 2^n} I_{n,k}, \quad n \ge 1$$



$$|I_{n,k}| \asymp \int_{T_{n,k}} \frac{dm_2(z)}{1 - |z|}$$

$$\operatorname{Ent} E = \infty \Leftrightarrow \sum_{I_{n,k} \cap E} |I_{n,k}| = \infty.$$

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Question: Is the weighted area condition sufficient for multiplying inside (GCN)?

Example. $\exists$ a singular S such that

$$\forall \delta < 1 \quad \int_{|S|<\delta} \frac{dm_2(z)}{1-|z|} < \infty$$

But $\nexists$ Blaschke product B such that $BS \in (GCN)$.

<u>Construction</u>

(in $\mathbb{C}_+$) μ singular measure on $\mathbb{R}$ w/compact support such that

(i) $\forall \delta > 0$, $\mathcal{A}(\Delta(S,\delta)) = \displaystyle\int_{\Delta(S,\delta)} \frac{dm_2(z)}{\operatorname{Im} z} < \infty$

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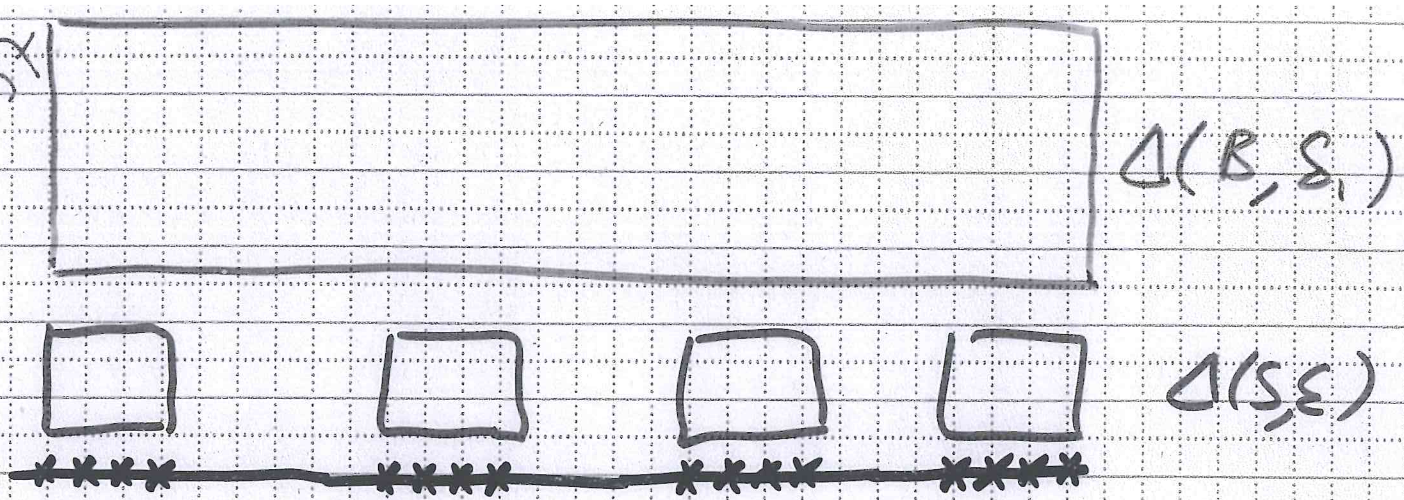
(ii) If $\varepsilon > 0$, B Blaschke product such that $\Delta(S,\varepsilon) \subset \mathcal{D}(Z(B), \tfrac{1}{2})$, then $\forall\ \delta_1 > 0$ we have

$$\mathcal{A}(\Delta(B,\delta_1)) = \infty.$$



- $E \subset \mathbb{T}$ closed, $|E| = 0$,
  is porous if
  $\forall$ arc $I \subset \mathbb{T}$ $\exists$ arc $J \subset I \setminus E$:
  $$|J| \asymp |I|$$

$\Leftrightarrow$ uniformly finite entropy w.r.t. Möbius transforms

$$\Leftrightarrow \sum_{\substack{J \text{ dyadic} \\ J \subset I,\, J \cap E \neq \emptyset}} |J| \lesssim |I|$$

Theorem (Nicolau) E porous, $\operatorname{supp} \mu \subset E \Rightarrow \exists\, B$: $BS_\mu \in (GCN)$

- Question: $\operatorname{Ent} E < \infty$, but E not porous.