On AAK-theory in weighted spaces.

Marcus Carlsson

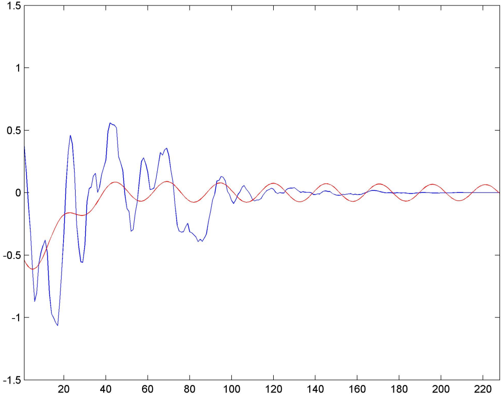
May 31, 2013

Part I:

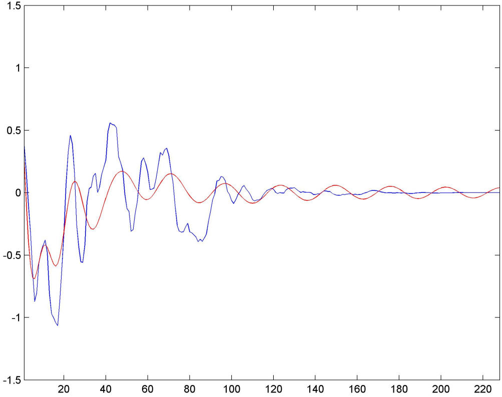
AAK-theory is very useful

Outline of an AAK-based algorithm

k=3:

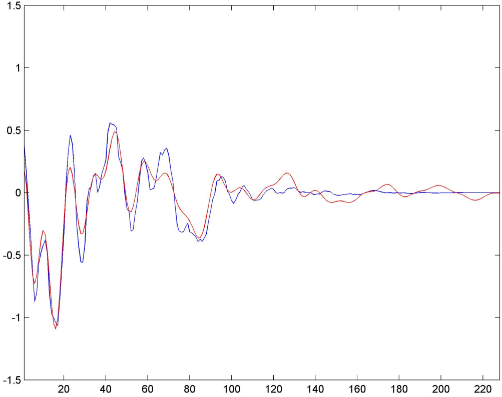


k=5:



Outline of an AAK-based algorithm

k=9:



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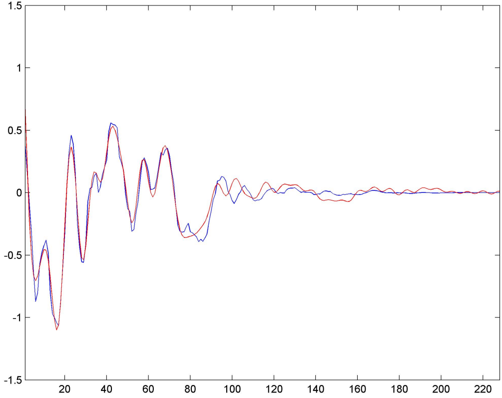
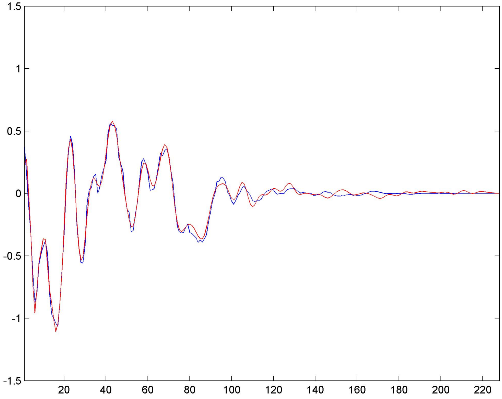
k=15:



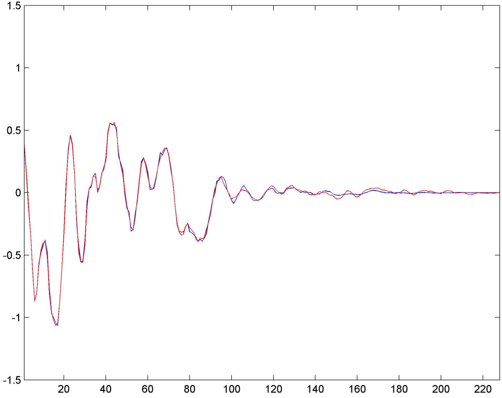
Outline of an AAK-based algorithm

k=19:



Outline of an AAK-based algorithm

k=29:



Outline of an AAK-based algorithm

1. You wish to approximate a function F on $\mathbb{R}^+$ by a linear combination of **few** exponential functions.
2. Consider the Hankel operator on $L^2(\mathbb{R}^+)$ with symbol F;

$$\Gamma_F(G)(x) = \int_0^\infty F(x+y)G(y)dy.$$

3. Compute singular values and vectors of Γ_F, call them $(\sigma_j)_{j=0}^\infty$ and $(u_j)_{j=0}^\infty$.

   (That is $\sigma_j^2 u_j = \Gamma_F^* \Gamma_F u_j$ and we set $\sigma_0 \geq \sigma_1 \ldots$)

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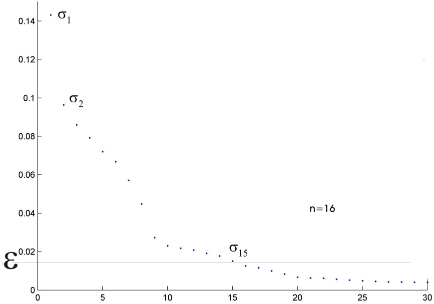
($F \longrightarrow \Gamma_F \longrightarrow$ Sing. value's σ_j & sing. vector's u_j.)

4 Pick a σ_k close to your desired error ε, with k as *small* as possible.



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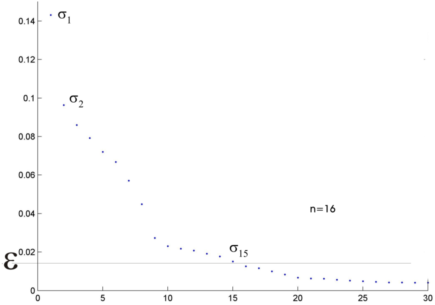

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Outline of an AAK-based algorithm

$(F \longrightarrow \Gamma_F \longrightarrow \sigma_k \approx \varepsilon \longrightarrow u_k \longrightarrow \check{u}_k \longrightarrow (z_m)_{m=1}^k.)$

6. *Adamyan-Arov-Krein:* There are coefficients $c_1, \ldots, c_n \in \mathbb{C}$ such that

$$F(x) \approx \sum_{m=1}^{k} c_m \mathrm{e}^{\mathrm{i} z_m x}$$

with error $\approx \varepsilon$.

Outline of an AAK-based algorithm

Pros:

- Completely non-linear approximation algorithm
- Complex frequencies
- Accuracy is chosen first

Cons:

- Set up not flexible, does not work with weights...

Or is there an AAK-type theory in weighted spaces?

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Part II:

AAK-theory in weighted spaces

Hankel operators in weighted spaces.

Let $w = (w_j)_{j=0}^{\infty}$ be a weight. Set

$$\ell_w^2 = \{(a_j)_{j=0}^{\infty} : \sum_{j=0}^{\infty} |a_j|^2 w_j < \infty\}.$$

Definition in this talk: A Hankel operator on ℓ_w^2 is an operator Γ_f whose matrix representation in the canonical basis looks like

$$\Gamma_f = \begin{pmatrix} f_0 & f_1 & f_2 & \cdots \\ f_1 & f_2 & \cdots & \cdots \\ f_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Equivalent definition: An operator Γ is Hankel if and only if $B\Gamma = \Gamma S$, where S is the forward shift operator and B is the backward shift operator.

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AAK-theory in the unweighted case.

Theorem 1 (AAK) Given a Hankel operator Γ_f on $\ell^2 (= \ell^2_1)$ and a singular value σ_k, there exists a Hankel operator Γ_g of rank k such that

$$\|\Gamma_f - \Gamma_g\| = \sigma_k.$$

Moreover, g is then a sum of k geometric sequences which can be explicitly found by computing the k zeroes of $\check{u}_k(z) \in H^2(\mathbb{D})$.

($u_k \in \ell^2$ is the k'th singular vector and $\check{u}_k(z) = \sum_{j=0}^{\infty} u_{k,j} z^j$)

Theorem 2 ("Equivalent" reformulation of AAK) There exists a shift-invariant subspace $\mathcal{M} \subset \ell^2$ of codimension k, such that

$$\|\Gamma_f|_{\mathcal{M}}\| = \sigma_k.$$

Moreover,
$\check{\mathcal{M}} = \{$the z-invariant subspace with the same zeroes as $\check{u}_k\}$.

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Here is what happens if we consider Γ_f on ℓ^2_w where w is not constant:

Theorem 1 seems to be false always.

Theorem 2 (first part) is true whenever w is increasing. (due to (S. Treil and A. Volberg 1994))

Theorem 2 (second part) is true whenever w is strictly increasing and "point evaluations on the boundary $\mathbb{T}$ are not bounded", i.e. whenever $\sum_{j=0}^{\infty} \frac{1}{w_j} = \infty$. (Carlsson 2009)

Conjecture: The second assumption is not necessary.

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Suppose $\|\Gamma_f\| = 1$.

$$\Gamma_f = \begin{pmatrix} f_0 & f_1 & f_2 & \dots \\ f_1 & f_2 & \cdot & \dots \\ f_2 & \cdot & \cdot & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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None.

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Related result: Commutant lifting:

$\exists f = (f_k)_{k=-\infty}^{\infty}$ such that $\|Ext_{\Gamma_f}\| = 1$ where $Ext_{\Gamma_f} : \ell^2_w(\mathbb{N}) \to \ell^2_w(\mathbb{Z})$ is "Hankel", i.e. looks like

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Spelled out, the main theorem reads:

Theorem Let the weight w be strictly increasing and satisfy

$$\sum_{j=0}^{\infty} \frac{1}{w_j} = \infty.$$

Let Γ be a Hankel operator on ℓ^2_w. Let σ_k/u_k be the kth singular value/vector. Then $\check{u}_k(z) = \sum_{j=0}^{\infty} u_{k,j} z^j$ has precisely k zeroes in $\mathbb{D}$. Moreover if the zeroes $\{z_j\}_{j=1}^{k}$ of $\check{u}_k$ are disjoint and

$$\check{\mathcal{M}} = \{h \in \check{\ell}^2_w : \ h(z_j) = 0 \text{ for } j = 1 \ldots, k\},$$

then

$$\|\Gamma|_{\mathcal{M}}\| = \sigma_k.$$