On AAK-theory in weighted spaces.

Marcus Carlsson

May 31, 2013
Part I:
AAK-theory is very useful
Outline of an AAK-based algorithm

$k=3:$
Outline of an AAK-based algorithm

\( k=5: \)

![Graph showing multiple lines with peaks and troughs over a range of x-values from 20 to 220.](image-url)
Outline of an AAK-based algorithm

$k=9$: 

![Graph showing a time series with oscillations and a linear trend](image-url)
Outline of an AAK-based algorithm

$k=15$: 

![Graph](image-url)
Outline of an AAK-based algorithm

$k=19$: 

![Graph showing data points and lines]
Outline of an AAK-based algorithm

1. You wish to approximate a function $F$ on $\mathbb{R}^+$ by a linear combination of few exponential functions.

2. Consider the Hankel operator on $L^2(\mathbb{R}^+)$ with symbol $F$;

$$\Gamma_F(G)(x) = \int_0^\infty F(x + y)G(y)dy.$$ 

3. Compute singular values and vectors of $\Gamma_F$, call them $(\sigma_j)_{j=0}^\infty$ and $(u_j)_{j=0}^\infty$.

(That is $\sigma_j^2 u_j = \Gamma_F^* \Gamma_F u_j$ and we set $\sigma_0 \geq \sigma_1$...)

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\[(F \rightarrow \Gamma_F \rightarrow \text{Sing. value's } \sigma_j \& \text{ sing. vector's } u_j.)\]

4 Pick a \( \sigma_k \) close to your desired error \( \varepsilon \), with \( k \) as small as possible.

5 \( \tilde{u}_k(z) \) is a \( H^2(\mathbb{C}^+) \) function, which according to AAK has precisely \( k \) zeroes. Compute these and call them \( z_1, \ldots, z_k \).
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(F \rightarrow \Gamma_F \rightarrow \sigma_k \approx \varepsilon \rightarrow u_k \rightarrow \tilde{u}_k \rightarrow (z_m)_{m=1}^k.)

6 *Adamyan-Arov-Krein:* There are coefficients $c_1, \ldots, c_n \in \mathbb{C}$ such that

$$F(x) \approx \sum_{m=1}^k c_m e^{i z_m x}$$

with error $\approx \varepsilon$. 
Outline of an AAK-based algorithm

Pros:
- Completely non-linear approximation algorithm
- Complex frequencies
- Accuracy is chosen first

Cons:
- Set up not flexible, does not work with weights...

Or is there an AAK-type theory in weighted spaces?
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Cons:
▶ Set up not flexible, does not work with weights...

Or is there an AAK-type theory in weighted spaces?
Part II: AAK-theory in weighted spaces
Let $w = (w_j)_{j=0}^\infty$ be a weight. Set

$$\ell^2_w = \{(a_j)_{j=0}^\infty : \sum_{j=0}^\infty |a_j|^2 w_j < \infty\}.$$ 

**Definition in this talk:** A Hankel operator on $\ell^2_w$ is an operator $\Gamma_f$ whose matrix representation in the canonical basis looks like

$$\Gamma_f = \begin{pmatrix} f_0 & f_1 & f_2 & \cdots \\ f_1 & f_2 & \cdots & \cdots \\ f_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

**Equivalent definition:** An operator $\Gamma$ is Hankel if and only if $B\Gamma = \Gamma S$, where $S$ is the forward shift operator and $B$ is the backward shift operator.
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**Equivalent definition:** An operator $\Gamma$ is Hankel if and only if $B \Gamma = \Gamma S$, where $S$ is the forward shift operator and $B$ is the backward shift operator.
Theorem 1 (AAK) Given a Hankel operator $\Gamma_f$ on $\ell^2(=\ell^1_1)$ and a singular value $\sigma_k$, there exists a Hankel operator $\Gamma_g$ of rank $k$ such that

$$\|\Gamma_f - \Gamma_g\| = \sigma_k.$$ 

Moreover, $g$ is then a sum of $k$ geometric sequences which can be explicitly found by computing the $k$ zeroes of $\check{u}_k(z) \in H^2(\mathbb{D})$.

\begin{align*}
\left(u_k \in \ell^2 \text{ is the } k\text{'th singular vector and } \check{u}_k(z) = \sum_{j=0}^{\infty} u_{k,j} z^j\right)
\end{align*}

Theorem 2 (“Equivalent” reformulation of AAK) There exists a shift-invariant subspace $\mathcal{M} \subset \ell^2$ of codimension $k$, such that

$$\|\Gamma_f|_{\mathcal{M}}\| = \sigma_k.$$ 

Moreover,

$\tilde{\mathcal{M}} = \{\text{the } z\text{-invariant subspace with the same zeroes as } \check{u}_k\}$. 
Theorem 1 (AAK) Given a Hankel operator $\Gamma_f$ on $\ell^2(=\ell_1^2)$ and a singular value $\sigma_k$, there exists a Hankel operator $\Gamma_g$ of rank $k$ such that

$$\|\Gamma_f - \Gamma_g\| = \sigma_k.$$ 

Moreover, $g$ is then a sum of $k$ geometric sequences which can be explicitly found by computing the $k$ zeroes of $\tilde{u}_k(z) \in H^2(\mathbb{D})$.

Theorem 2 (“Equivalent” reformulation of AAK) There exists a shift-invariant subspace $M \subset \ell^2$ of codimension $k$, such that

$$\|\Gamma_f|_M\| = \sigma_k.$$ 

Moreover, $M = \{\text{the } z\text{-invariant subspace with the same zeroes as } \tilde{u}_k\}$.
Here is what happens if we consider $\Gamma_f$ on $\ell^2_w$ where $w$ is not constant:

**Theorem 1** seems to be false always.

Theorem 2 (first part) is true whenever $w$ is increasing.

(due to (S. Treil and A. Volberg 1994))

Theorem 2 (second part) is true whenever $w$ is strictly increasing and “point evaluations on the boundary $\mathbb{T}$ are not bounded”, i.e. whenever $\sum_{j=0}^{\infty} \frac{1}{w_j} = \infty$. (Carlsson 2009)

**Conjecture:** The second assumption is not necessary.
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Conjecture: The second assumption is not necessary.
None.
Reason: Theorem 1 fails.
Related result: Commutant lifting:
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Suppose $\|\Gamma_f\| = 1$.

$$
\Gamma_f = \begin{pmatrix}
    f_0 & f_1 & f_2 & \cdots \\
    f_1 & f_2 & \cdot & \cdots \\
    f_2 & \cdot & \cdot & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$
Applications?

None.

Reason: Theorem 1 fails.

Related result: Commutant lifting:

\[ \exists f = (f_k)_{k=-\infty}^{\infty} \text{ such that } \| Ext_{\Gamma_f} \| = 1 \text{ where } \]

\[ Ext_{\Gamma_f} : \ell_w^2(\mathbb{N}) \to \ell_w^2(\mathbb{Z}) \text{ is "Hankel"}, \text{ i.e. looks like} \]

\[ Ext_{\Gamma_f} = \begin{pmatrix}
    \vdots & \vdots & \vdots & \ddots \\
    f_{-2} & f_{-1} & f_0 & \ldots \\
    f_{-1} & f_0 & f_1 & \ldots \\
    f_0 & f_1 & f_2 & \ldots \\
    f_1 & f_2 & \ddots & \ldots \\
    f_2 & \ddots & \ddots & \ddots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \]
Applications?

None.

Reason: Theorem 1 fails.

Related result: Commutant lifting:

However, \( \mathcal{A} f = (f_k)_{k=-\infty}^{\infty} \) such that \( \|\text{Ext}_{\Gamma_f}\| = 1 \) where \( \text{Ext}_{\Gamma_f} : \ell^2_w(\mathbb{Z}) \to \ell^2_w(\mathbb{Z}) \) is "Hankel", i.e. looks like

\[
\text{Ext}_{\Gamma_f} = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\vdots & f_{-2} & f_{-1} & f_0 & \cdot \\
\vdots & f_{-2} & f_{-1} & f_0 & f_1 & \cdot \\
\vdots & f_{-1} & f_0 & f_1 & f_2 & \cdot \\
\vdots & f_0 & f_1 & f_2 & \cdot & \cdot \\
\vdots & f_1 & f_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\]
Theorem  Let the weight $w$ be strictly increasing and satisfy

$$\sum_{j=0}^{\infty} \frac{1}{w_j} = \infty.$$ 

Let $\Gamma$ be a Hankel operator on $\ell^2_w$. Let $\sigma_k/u_k$ be the $k$th singular value/vector. Then $\check{u}_k(z) = \sum_{j=0}^{\infty} u_k_j z^j$ has precisely $k$ zeroes in $\mathbb{D}$. Moreover if the zeroes $\{z_j\}_{j=1}^{k}$ of $\check{u}_k$ are disjoint and

$$\tilde{\mathcal{M}} = \{ h \in \ell^2_w : h(z_j) = 0 \text{ for } j = 1 \ldots, k \},$$

then

$$\|\Gamma|_{\tilde{\mathcal{M}}}\| = \sigma_k.$$