Rota's Universal Operators and Invariant Subspaces in Hilbert Spaces

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Speaker thanks the Departamento Análisis Matemático, Univ. Complutense de Madrid for hospitality during summer 2011 and fall 2012, and also thanks IUPUI for a sabbatical during 2012-13 that made this work possible.

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Today, I want to present what is new and interesting $\cdots$ and true!

The Hardy Hilbert space on the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is:

$$H^2 = \{h \text{ analytic in } \mathbb{D} : h(z) = \sum_{n=0}^{\infty} a_n z^n \;\text{ with }\; \|h\|^2 = \sum |a_n|^2 < \infty\}$$

where for h and g in H^2, we have $\langle h, g \rangle = \sum a_n \overline{b_n}$

Writing h as a Fourier series: $h \sim \sum_{n=0}^{\infty} a_n e^{in\theta}$

$H^2(\mathbb{D})$ is closed subspace of $L^2(\partial\mathbb{D})$ consisting of h with $a_n = 0$ for $n < 0$.

Second definition:

$$H^2(\mathbb{D}) = \{h \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \int_0^{2\pi} |h(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty\}$$

Write K_α for function in H^2 that gives $\langle h, K_\alpha \rangle = h(\alpha)$:

$$K_\alpha(z) = (1 - \overline{\alpha} z)^{-1} \text{ for } \alpha \text{ in } \mathbb{D}.$$

Consider four types of operators on H^2:

For f in $L^\infty(\partial\mathbb{D})$, *Toeplitz operator* T_f is operator given by $T_f h = P_+ f h$

where P_+ is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2

For ψ a bounded analytic map of $\mathbb{D}$ into the complex plane,

the *analytic Toeplitz operator* T_ψ is

$$(T_\psi h)(z) = \psi(z)h(z) \quad \text{for } h \text{ in } H^2$$

Note: for ψ in H^∞, $\quad P_+\psi h = \psi h$

For φ an analytic map of $\mathbb{D}$ into itself, the *composition operator* C_φ is

$$(C_\varphi h)(z) = h(\varphi(z)) \quad \text{for } h \text{ in } H^2$$

and for ψ in H^∞ and φ an analytic map of $\mathbb{D}$ into itself,

the *weighted composition operator* $W_{\psi,\varphi} = T_\psi C_\varphi$ is

$$(W_{\psi,\varphi} h)(z) = \psi(z)h(\varphi(z)) \quad \text{for } h \text{ in } H^2$$

Some terminology:

If A is bounded operator on Banach space $\mathcal{X}$, a closed subspace M, of $\mathcal{X}$ is

a *non-trivial invariant subspace for A*

if for all v in M, the vector Av is also in M and $(0) \neq M \neq \mathcal{X}$.

If M is a non-trivial invariant subspace for A, then M is

a *hyperinvariant subspace for A*

if M is invariant subspace for all operators, B, satisfying $AB = BA$.

The *Invariant Subspace Questions* are:

- Does every bounded operator on a Banach space have a non-trivial

  invariant subspace? $\cdots$ hyperinvariant subspace?

Some history:

- Restrict attention to $A \neq \lambda I$ and $2 \leq \dim(\mathcal{X})$, with $\mathcal{X}$ separable
- von Neumann ('30's), Aronszajn & Smith ('54) compacts
- Lomonosov ('73):

  A bounded linear operator T, not a multiple of the identity,

  that commutes with a nonzero compact operator,

  has a non-trivial hyperinvariant closed subspace.

  This implies that *An operator that commutes with*

  a non-scalar operator that commutes with a non-zero compact

  has a non-trivial invariant subspace.

- Lomonosov did *not* solve ISP: Hadwin, Nordgren, Radjavi, Rosenthal('80)
- Enflo ('75/'87), Read ('85)
- Scott Brown ('78)

Rota's Universal Operators:

Defn: Let $\mathcal{X}$ be a Banach space, let U be a bounded operator on $\mathcal{X}$, and let $\mathcal{B}(\mathcal{X})$ be the algebra of bounded operators on $\mathcal{X}$.

We say U is *universal for* $\mathcal{X}$ if for each non-zero bounded operator A on $\mathcal{X}$, there is an invariant subspace M for U and a non-zero number λ such that λA is similar to $U|_M$.

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Rota proved in 1960 that if $\mathcal{X}$ is a separable, infinite dimensional Hilbert space, there are universal operators on $\mathcal{X}$!

Theorem (Caradus (1969))

If $\mathcal{H}$ is separable Hilbert space and U is bounded operator on $\mathcal{H}$ such that:

1. The null space of U is infinite dimensional.

2. The range of U is $\mathcal{H}$.

then U is universal for $\mathcal{H}$.

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We give a universal operator commuting with a compact operator.

Some previously known Universal Operators (in sense of Rota):

Best Known: adjoint of a unilateral shift of infinite multiplicity:

Suppose S is analytic Toeplitz operator whose symbol is singular inner function or infinite Blaschke product, then S^* is a universal operator.

Such an operator can be represented as a block matrix on $\mathcal{H} = \oplus_{k=0}^{\infty} \mathcal{W}$, where $\mathcal{W}$ is the "wandering subspace" associated with S.

The block matrix representing S^* is upper triangular and has the identity on $\mathcal{W}$ on the super-diagonal:

$$S^* \sim \begin{pmatrix} 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ 0 & 0 & 0 & I & \cdots \\ & & & \ddots & \end{pmatrix}$$

Easy computation shows that every operator commuting with S^* has form

$$A \sim \begin{pmatrix} A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ 0 & A_0 & A_{-1} & A_{-2} & \cdots \\ 0 & 0 & A_0 & A_{-1} & \cdots \\ & & & \ddots & \end{pmatrix}$$

that is, upper triangular block Toeplitz matrix, that is, an upper triangular block matrix whose entries on each diagonal are the same operator on the infinite dimensional Hilbert space $\mathcal{W}$.

Because every block in such a matrix occurs infinitely often, we easily see that only compact operator that commutes with universal operator S^* is 0.

Some previously known Universal Operators (in sense of Rota):

Also well known (Nordgren, Rosenthal, Wintrobe ('84,'87)):

If φ is an automorphism of $\mathbb{D}$ with fixed points ± 1 and Denjoy-Wolff point 1,

that is, $\varphi(z) = \dfrac{z+s}{1+sz}$ for $0 < s < 1$,

then a translate of the composition operator C_φ is a universal operator.

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In C.'s thesis ('76): The analytic Toeplitz operators S and T_ψ *DO NOT* commute with non-trivial compact operators.

Also proved: *IF* an analytic Toeplitz operator commutes with a non-trivial compact, then the compact operator is quasi-nilpotent.

Our Goal Today:

There is an analytic Toeplitz operator that commutes with a non-trivial compact operator and the adjoint of the Toeplitz operator is a universal operator in the sense of Rota.

Lemma.

For φ and ψ in H^∞ and J an analytic map of the unit disk into itself, the analytic Toeplitz operator T_φ commutes with the composition operator C_J

or

T_φ commutes with the weighted composition operator $W_{\psi,J}$

if and only if

$\varphi \circ J = \varphi$.

Proof:

We will skip the easy proof.

Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z^2 > -1 \text{ and } \operatorname{Re} z < 0\}$, region in second quadrant above branch of the hyperbola $2xy = -1$.

Let σ be the Riemann map of $\mathbb{D}$ onto Ω defined by

$$\sigma(z) = \frac{-1+i}{\sqrt{z+1}}$$

branch of $\sqrt{\cdot}$ on the halfplane $\{z : \operatorname{Re} z > 0\}$ satisfies $\sqrt{1} = 1$.

Notice that $\sigma(1) = (-1+i)/\sqrt{2}$, $\sigma(0) = -1+i$, and $\sigma(-1) = \infty$.

We define φ on the unit disk by

$$\varphi(z) = e^{\sigma(z)} - e^{\sigma(0)} = e^{\sigma(z)} - e^{-1+i}$$

Helpful to point out some of the properties of σ, e^{σ}, and φ. Use the set $\Gamma = \{e^{i\theta} : -\pi < \theta < \pi\}$, the unit circle except -1, in this description.

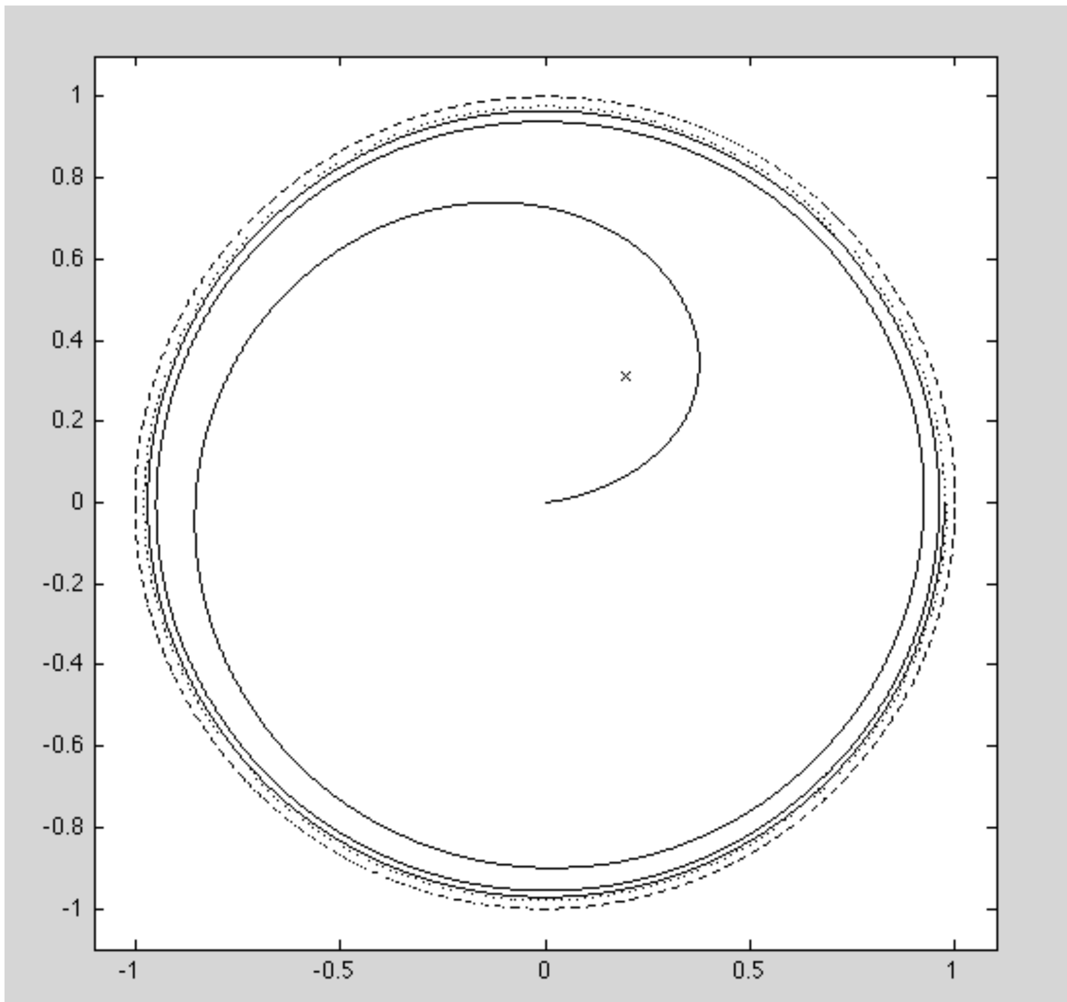
[1] $\Omega = \sigma(\mathbb{D})$ is the region in the second quadrant of the complex plane above the branch of the hyperbola $2xy = -1$ and this branch is $\sigma(\Gamma)$. Moreover, $\sigma(0)$ is not on the curve $\sigma(\Gamma)$.

[1] $\Omega = \sigma(\mathbb{D})$ is the region above the branch of the hyperbola $2xy = -1$.

[2] The function e^{σ} maps curve Γ onto curve spiraling out from origin to $\partial\mathbb{D}$.

Circle of radius r intersects curve $e^{\sigma(\Gamma)}$ in exactly one point.

Closure $e^{\sigma(\Gamma)}$ is the set $\{0\} \cup e^{\sigma(\Gamma)} \cup \partial\mathbb{D}$ and distance $e^{\sigma(0)}$ to $e^{\sigma(\Gamma)} > 0$.



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Lemma.

If f is a function in $H^{\infty}(\mathbb{D})$ and there is $\ell > 0$ so that $|f(e^{i\theta})| \geq \ell$ almost everywhere on the unit circle, then $1/f$ is in $L^{\infty}(\partial\mathbb{D})$ and the (non-analytic) Toeplitz operator $T_{1/f}$ is a left inverse for the analytic Toeplitz operator T_f.

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Corollary.

The analytic Toeplitz operator T_φ has a left inverse.

Corollary.

Toeplitz operator T_φ^* has right inverse and T_φ^* maps $H^2(\mathbb{D})$ onto itself.

The following is our first main Theorem:

Theorem.

The Toeplitz operator T_φ^* is universal for $H^2(\mathbb{D})$.

Proof:

We use the Theorem of Caradus to establish the result.

The last Corollary shows that the range of T_φ^* is all of $H^2(\mathbb{D})$.

For n a non-negative integer, let $z_n = \sigma^{-1}(-1 + i + 2n\pi i)$.

For each n, the vector K_{z_n} is in the nullspace of T_φ^*,

so the nullspace of T_φ^* is infinite dimensional.

Therefore, by Caradus' Theorem, T_φ^* is a universal operator for $H^2(\mathbb{D})$.

Let J be the analytic map of the unit disk into itself given by

$$J(z) = \sigma^{-1}(\sigma(z) + 2\pi i)$$

From this definition, an easy calculation shows that $\varphi \circ J = \varphi$.

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We will show the image $J(\mathbb{D})$ is a convex set in $\mathbb{D}$.

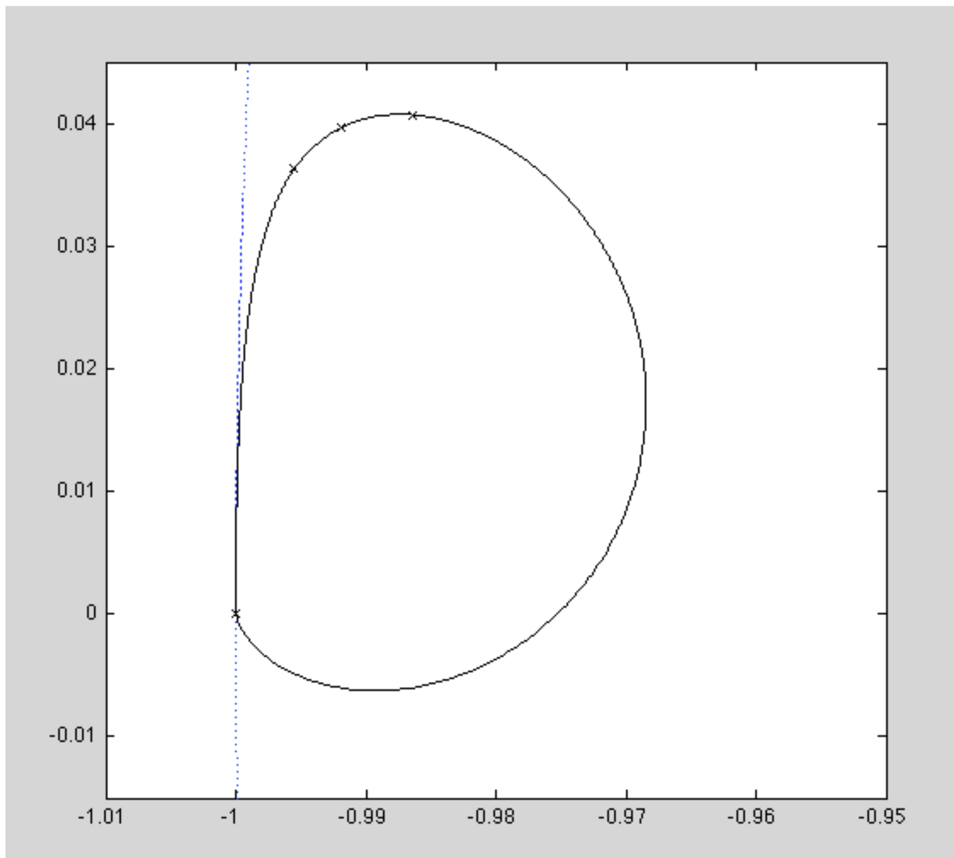

Figure 1: The set $J(\partial\mathbb{D})$ with $J(-1) = -1$, $J(-i)$, $J(1)$, and $J(i)$.

Recall that σ is defined by

$$\zeta = \sigma(z) = \frac{-1+i}{\sqrt{z+1}}$$

which means σ^{-1} is given by

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Let H be branch of the hyperbola $2x(y-2\pi) = -1$ in 2^{nd} quadrant.

Since J is given by $J(z) = \sigma^{-1}(\sigma(z) + 2\pi i)$

showing that $J(\mathbb{D})$ is convex

means showing that $\sigma^{-1}(\text{region above hyperbola } H\ )$ is convex.

There is a standard criterion for deciding whether the image of the unit disk under a univalent map is convex $\cdots$ this works, but it's tedious.

Instead, we'll use the fact that the non-empty intersection of convex sets is convex and exhibit $J(\mathbb{D})$ as such an intersection.

Lemma.

For each point, p, on the hyperbola H, there is a unique circle, Z_p, passing through 0 and tangent to the hyperbola at p.

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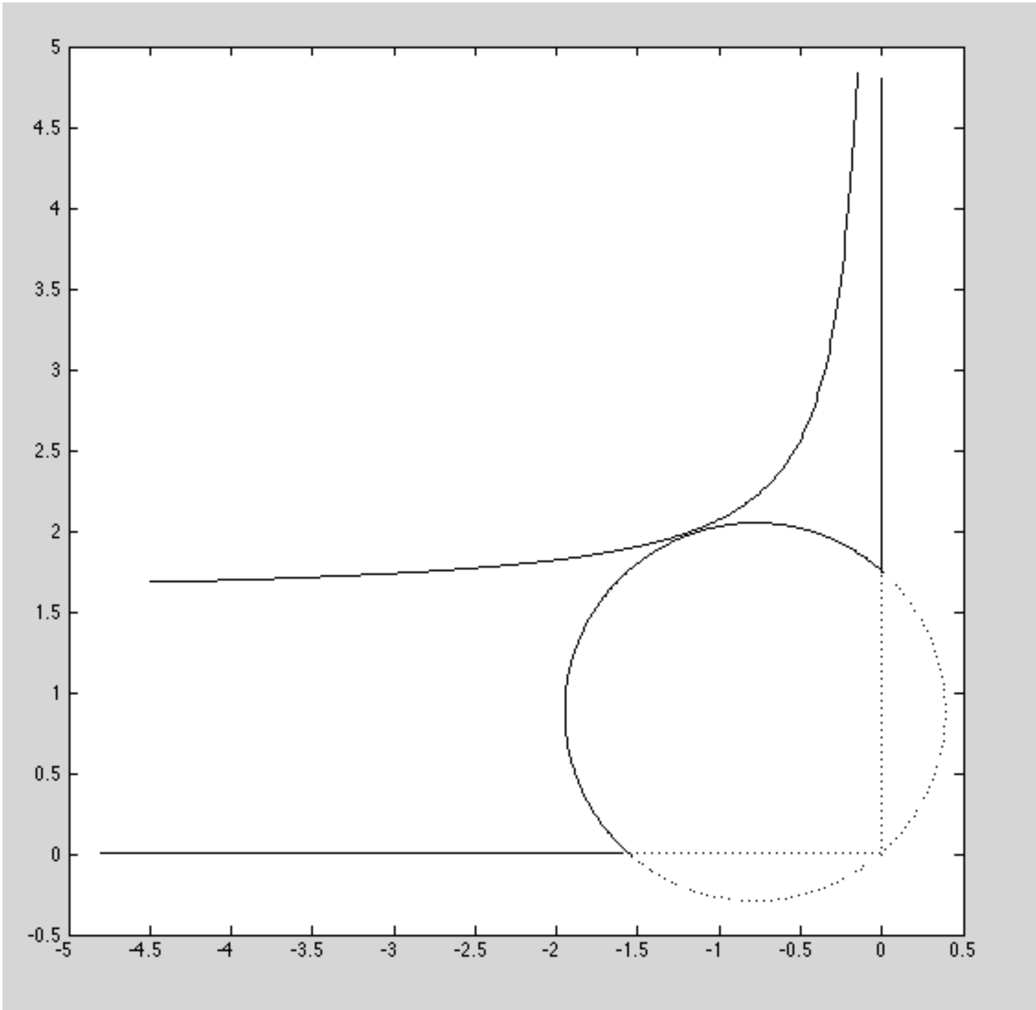

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Lemma.

$$\bigcap_{p \in H} \Delta_p \quad \text{is the region above the hyperbola } H.$$

For each p in H, the set $\{\zeta^{-1} : \zeta \in \Delta_p\}$ is the interior of a *Euclidean triangle*(!!) (or a half strip if 0 is a vertex of Δ_p) two of whose sides lie on the negative real and negative imaginary axes.

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For each p in H, the set $\{\zeta^{-2} : \zeta \in \Delta_p\}$ is the interior of a *parabolic section*(!!) in the upper half plane whose straight boundary is on the real axis. That is, for each p in H, the set $\{\zeta^{-2} : \zeta \in \Delta_p\}$ is a convex set.

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Since $\sigma^{-1}(\Delta_p)$ is a convex set for each p in H and

$$\bigcap_{p \in H} \sigma^{-1}(\Delta_p) = \sigma^{-1}\left(\bigcap_{p \in H} \Delta_p\right) = J(\mathbb{D})$$

we see that $J(\mathbb{D})$ is convex.

Because $J(\mathbb{D})$ is convex, the polynomials in J are weak-star dense in H^∞, and C_J has dense range.

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The function $\psi(z) = (z+1)/2$ is an outer function, so T_ψ has dense range.

Conclude $W_{\psi,J} = T_\psi C_J$ has dense range, so $W^*_{\psi,J}$ is injective.

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Because ψ is continuous, $\psi(-1) = 0$, and -1 is Denjoy-Wolff point of J, work of Gunatillake (2007) or C. (1980) implies $W_{\psi,J}$ is compact.

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Main Theorem.

The operator $W_{\psi,J}^*$ is an injective, compact operator that commutes with the universal operator T_φ^*.

PLAN:

Use this result to show that every operator on a separable Hilbert space has a non-trivial invariant subspace by:

Because T_φ^* is universal, if A_0 is a given operator on a separable Hilbert space $\mathcal{H}$, then there is an invariant subspace M for T_φ^* such that A_0 is similar to $T_\varphi^*\big|_M$, which we will call A.

Obviously, if $W_{\psi,J}^*$ ALSO has M as an invariant subspace,

then $W_{\psi,J}^*\big|_M$ is a non-zero compact that commutes with A.

But, we know that this cannot *always* be true!

One result that we were hopeful about was

Corollary.

If A is any bounded linear operator on a separable infinite dimensional Hilbert space, $\mathcal{H}$, there is a complex number μ and non-zero compact operators, K_1 and K_2, on $\mathcal{H}$ such that $K_1(A + \mu I) = (A + \mu I)K_2$.

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But, there are trivial examples of this: If A is any invertible operator and K is any compact operator on the same space, then $K_1 A = A(A^{-1}K_1 A)$ which can be written as $K_1 A = AK_2$ where $K_2 = (A^{-1}K_1 A)$

THANK YOU!