Realizations via preorders

Michael Dritschel

24 May 2013

First realization theorem

Theorem 1.
Suppose $\varphi : \mathbb{D} \to \mathbb{C}$. The following are equivalent.

(MB) φ *is in the Schur-Agler class; that is, if k is the Szegő kernel on $\mathbb{D}$ then the kernel $([1] - f^* f) \star k$ is positive;*

(AD) *There exists a positive kernel Γ such that* $[1] - \varphi^* \varphi = \Gamma \star ([1] - Z^* Z)$;

(TF) *There is a unitary colligation Σ so that $\varphi = W_\Sigma$;*

(vN) *For each strict contraction T on a Hilbert space $\mathcal{H}$,*

$$\|\varphi(T)\| \leq 1.$$

(That is, $\|\pi(\varphi)\| \leq 1$ for each continuous unital representation π of $H^\infty(\mathbb{D})$ which is strictly contractive on the coordinate function.)

First realization theorem

Theorem 1.
Suppose $\varphi : \mathbb{D} \to \mathbb{C}$. The following are equivalent.

(MB) *φ is in the Schur-Agler class; that is, if k is the Szegő kernel on $\mathbb{D}$ then the kernel $([1] - f^* f) \star k$ is positive;*

(AD) *There exists a positive kernel Γ such that $[1] - \varphi^* \varphi = \Gamma \star ([1] - Z^* Z)$;*

(TF) *There is a unitary colligation Σ so that $\varphi = W_\Sigma$;*

(vN) *For each strict contraction T on a Hilbert space $\mathcal{H}$,*

$$\|\varphi(T)\| \leq 1.$$

(That is, $\|\pi(\varphi)\| \leq 1$ for each continuous unital representation π of $H^\infty(\mathbb{D})$ which is strictly contractive on the coordinate function.)

Note that if $\varphi = D + CZ(I - AZ)^{-1}B$,

$$\varphi(T) = D \otimes I + (C \otimes T)((I \otimes I) - (A \otimes T))^{-1}(B \otimes I),$$

defines a representation of $H^\infty(\mathbb{D})$ by $\pi(\varphi) = \varphi(T)$.

Agler's realization theorem on the polydisk

Let $\mathcal{K}$ be the collection of all positive k on $\mathbb{D}^d$ such that for $j = 1, \ldots, d$,

$$([1] - Z_j^* Z_j) \star k \geq 0 \qquad \text{on } \mathbb{D}^d,$$

$Z_j(z) = Z_j((z_1, \ldots, z_d)) = z_j$.

Agler's realization theorem on the polydisk

Let $\mathcal{K}$ be the collection of all positive k on $\mathbb{D}^d$ such that for $j = 1, \ldots, d$,

$$([1] - Z_j^* Z_j) \star k \geq 0 \qquad \text{on } \mathbb{D}^d,$$

$Z_j(z) = Z_j((z_1, \ldots, z_d)) = z_j$.

Set $H^\infty(\mathcal{K})$ to be all those functions φ on $\mathbb{D}^d$ for which there is some $C > 0$ such that then $(C[1] - \varphi^* \varphi) \star k \geq 0$ for all $k \in \mathcal{K}$ (the infimum of such C gives a norm making $H^\infty(\mathcal{K})$ a Banach algebra).

Agler's realization theorem on the polydisk

Let $\mathcal{K}$ be the collection of all positive k on $\mathbb{D}^d$ such that for $j = 1, \ldots, d$,

$$([1] - Z_j^* Z_j) \star k \geq 0 \qquad \text{on } \mathbb{D}^d,$$

$Z_j(z) = Z_j((z_1, \ldots, z_d)) = z_j$.

Set $H^\infty(\mathcal{K})$ to be all those functions φ on $\mathbb{D}^d$ for which there is some $C > 0$ such that then $(C[1] - \varphi^* \varphi) \star k \geq 0$ for all $k \in \mathcal{K}$ (the infimum of such C gives a norm making $H^\infty(\mathcal{K})$ a Banach algebra).

The *Schur-Agler class* $H_1^\infty(\mathcal{K})$ is the unit ball of $H^\infty(\mathcal{K})$.

Agler's realization theorem on the polydisk

Let $\mathcal{K}$ be the collection of all positive k on $\mathbb{D}^d$ such that for $j = 1, \ldots, d$,

$$([1] - Z_j^* Z_j) \star k \geq 0 \qquad \text{on } \mathbb{D}^d,$$

$Z_j(z) = Z_j((z_1, \ldots, z_d)) = z_j$.

Set $H^\infty(\mathcal{K})$ to be all those functions φ on $\mathbb{D}^d$ for which there is some $C > 0$ such that then $(C[1] - \varphi^* \varphi) \star k \geq 0$ for all $k \in \mathcal{K}$ (the infimum of such C gives a norm making $H^\infty(\mathcal{K})$ a Banach algebra).

The *Schur-Agler class* $H_1^\infty(\mathcal{K})$ is the unit ball of $H^\infty(\mathcal{K})$.

Theorem 2 (Agler).

Suppose $\varphi : \mathbb{D}^d \to \mathbb{C}$. *The following are equivalent.*

(MB) $\varphi \in H_1^\infty(\mathcal{K})$;

(AD) *There exist positive kernels* Γ_j *such that* $[1] - \varphi^* \varphi = \sum_1^d \Gamma_j \star ([1] - Z_j^* Z_j)$;

(TF) *There is a unitary colligation* Σ *so that* $\varphi = W_\Sigma$; *and*

(vN) *For each tuple* $T = (T_1, \ldots, T_d)$ *of commuting strict contractions on a Hilbert space* $\mathcal{H}$,

$$\|\varphi(T_1, \ldots, T_d)\| \leq 1.$$

(That is, $\|\pi(\varphi)\| \leq 1$ *for each continuous unital representation* π *of* $H^\infty(\mathcal{K})$ *which is strictly contractive on the coordinate functions.)*

Realization on the polydisk, continued

In the transfer function representation,

$$Z(z) = \sum_j P_j Z_j(z),$$

where the P_js are orthogonal projections summing to the identity.

In the transfer function representation,

$$Z(z) = \sum_j P_j Z_j(z),$$

where the P_js are orthogonal projections summing to the identity.

For $d > 2$, the Schur-Agler class is strictly contained within the unit ball of $H^\infty(\mathbb{D}^d)$.

Realization on the polydisk, continued

In the transfer function representation,

$$Z(z) = \sum_j P_j Z_j(z),$$

where the P_js are orthogonal projections summing to the identity.

For $d > 2$, the Schur-Agler class is strictly contained within the unit ball of $H^\infty(\mathbb{D}^d)$.

The Big Question

Can we realize the rest of $H_1^\infty(\mathbb{D}^d)$?

A collection Ψ of functions on a set X is a collection of *test functions* provided,

(i) For each $x \in X$,

$$\sup\{|\psi(x)| : \psi \in \Psi\} < 1; \text{ and}$$

(ii) for each finite set F, the unital algebra generated by $\Psi|_F$ is all of $P(F)$ (so $\Psi|_F$ separates the points of F).

Test functions

A collection Ψ of functions on a set X is a collection of *test functions* provided,

(i) For each $x \in X$,

$$\sup\{|\psi(x)| : \psi \in \Psi\} < 1; \text{ and}$$

(ii) for each finite set F, the unital algebra generated by $\Psi|_F$ is all of $P(F)$ (so $\Psi|_F$ separates the points of F).

We write $\mathcal{K}_\Psi$ for the collection of kernels k such that

$$([1] - \psi\psi^*) \star k \geq 0 \quad \text{for all } \psi \in \Psi.$$

Test functions

A collection Ψ of functions on a set X is a collection of *test functions* provided,

(i) For each $x \in X$,

$$\sup\{|\psi(x)| : \psi \in \Psi\} < 1; \text{ and}$$

(ii) for each finite set F, the unital algebra generated by $\Psi|_F$ is all of $P(F)$ (so $\Psi|_F$ separates the points of F).

We write $\mathcal{K}_\Psi$ for the collection of kernels k such that

$$([1] - \psi\psi^*) \star k \geq 0 \quad \text{for all } \psi \in \Psi.$$

Define $H^\infty(\mathcal{K}_\Psi)$ to be those functions f for which there is a $C < \infty$ such that

$$\big((C^2[1] - \varphi\,\varphi^*) \star k\big) \quad \text{for all } k \in \mathcal{K}_\Psi.$$

is a positive kernel for all $k \in \mathcal{K}_\Psi$. The *Schur-Agler class* $H_1^\infty(\mathcal{K}_\Psi)$ are all functions for which we can choose $C \leq 1$.

More on test functions

Suppose $\Psi = \{\psi_1, \ldots, \psi_d\}$ is a finite collection of test functions and P_js are orthogonal projections summing to the identity on $\mathcal{E}$.

More on test functions

Suppose $\Psi = \{\psi_1, \ldots, \psi_d\}$ is a finite collection of test functions and P_js are orthogonal projections summing to the identity on $\mathcal{E}$.

A unital representation $\rho : C(\Psi) \to B(\mathcal{E})$ has the form $\rho(f) = \sum_{j=1}^{N} P_j\, f(\psi_j)$.

More on test functions

Suppose $\Psi = \{\psi_1, \ldots, \psi_d\}$ is a finite collection of test functions and P_js are orthogonal projections summing to the identity on $\mathcal{E}$.

A unital representation $\rho : C(\Psi) \to B(\mathcal{E})$ has the form $\rho(f) = \sum_{j=1}^{N} P_j\, f(\psi_j)$.

Set $Z(x) = \sum_{j=1}^{N} P_j \psi_j(x)$ — so $Z(x) = \rho(E(x))$, where $E(x) \in C(\Psi)$ is evaluation at x.

More on test functions

Suppose $\Psi = \{\psi_1, \ldots, \psi_d\}$ is a finite collection of test functions and P_js are orthogonal projections summing to the identity on $\mathcal{E}$.

A unital representation $\rho : C(\Psi) \to B(\mathcal{E})$ has the form $\rho(f) = \sum_{j=1}^N P_j\, f(\psi_j)$.

Set $Z(x) = \sum_{j=1}^N P_j \psi_j(x)$ — so $Z(x) = \rho(E(x))$, where $E(x) \in C(\Psi)$ is evaluation at x.

The Agler decomposition can be phrased as

$$[1] - \varphi\, \varphi^* = \Gamma \star ([1] - E^* E), \qquad \Gamma : X \times X \to C(\Psi)^* \text{ positive.}$$

More on test functions

Suppose $\Psi = \{\psi_1, \dots, \psi_d\}$ is a finite collection of test functions and P_js are orthogonal projections summing to the identity on $\mathcal{E}$.

A unital representation $\rho : C(\Psi) \to B(\mathcal{E})$ has the form $\rho(f) = \sum_{j=1}^N P_j\, f(\psi_j)$.

Set $Z(x) = \sum_{j=1}^N P_j \psi_j(x)$ — so $Z(x) = \rho(E(x))$, where $E(x) \in C(\Psi)$ is evaluation at x.

The Agler decomposition can be phrased as

$$[1] - \varphi\,\varphi^* = \Gamma \star ([1] - E^* E), \qquad \Gamma : X \times X \to C(\Psi)^* \text{ positive.}$$

A positive kernel Γ has a Kolmogorov decomposition $\Gamma(x, y) = \gamma^*(y)\gamma(x)$, where $\gamma : C(\Psi)^* \to \mathcal{E}$ for some Hilbert space $\mathcal{E}$, and a GNS-type construction gives a repersentation $\rho : C(\Psi) \to \mathcal{B}(\mathcal{H})$ with $\rho(a)\gamma(x)b = \gamma(x)ab$.

More on test functions

Suppose $\Psi = \{\psi_1, \ldots, \psi_d\}$ is a finite collection of test functions and P_js are orthogonal projections summing to the identity on $\mathcal{E}$.

A unital representation $\rho : C(\Psi) \to B(\mathcal{E})$ has the form $\rho(f) = \sum_{j=1}^{N} P_j\, f(\psi_j)$.

Set $Z(x) = \sum_{j=1}^{N} P_j \psi_j(x)$ — so $Z(x) = \rho(E(x))$, where $E(x) \in C(\Psi)$ is evaluation at x.

The Agler decomposition can be phrased as

$$[1] - \varphi\,\varphi^* = \Gamma \star ([1] - E^* E), \qquad \Gamma : X \times X \to C(\Psi)^* \text{ positive.}$$

A positive kernel Γ has a Kolmogorov decomposition $\Gamma(x, y) = \gamma^*(y)\gamma(x)$, where $\gamma : C(\Psi)^* \to \mathcal{E}$ for some Hilbert space $\mathcal{E}$, and a GNS-type construction gives a repersentation $\rho : C(\Psi) \to \mathcal{B}(\mathcal{H})$ with $\rho(a)\gamma(x)b = \gamma(x)ab$.

So the Agler decomposition becomes

$$[1] - \varphi\,\varphi^* = \gamma^* \star ([1] - Z^* Z) \star \gamma.$$

More on test functions

Suppose $\Psi = \{\psi_1, \dots, \psi_d\}$ is a finite collection of test functions and P_js are orthogonal projections summing to the identity on $\mathcal{E}$.

A unital representation $\rho : C(\Psi) \to B(\mathcal{E})$ has the form $\rho(f) = \sum_{j=1}^{N} P_j\, f(\psi_j)$.

Set $Z(x) = \sum_{j=1}^{N} P_j \psi_j(x)$ — so $Z(x) = \rho(E(x))$, where $E(x) \in C(\Psi)$ is evaluation at x.

The Agler decomposition can be phrased as

$$[1] - \varphi\,\varphi^* = \Gamma \star ([1] - E^* E), \qquad \Gamma : X \times X \to C(\Psi)^* \text{ positive.}$$

A positive kernel Γ has a Kolmogorov decomposition $\Gamma(x, y) = \gamma^*(y)\gamma(x)$, where $\gamma : C(\Psi)^* \to \mathcal{E}$ for some Hilbert space $\mathcal{E}$, and a GNS-type construction gives a repersentation $\rho : C(\Psi) \to \mathcal{B}(\mathcal{H})$ with $\rho(a)\gamma(x)b = \gamma(x)ab$.

So the Agler decomposition becomes

$$[1] - \varphi\,\varphi^* = \gamma^* \star ([1] - Z^* Z) \star \gamma.$$

Given a representation π of $H^\infty(\mathcal{K}_\Psi)$ in $B(\mathcal{H})$, it is natural to define $\pi(Z) = \sum_{j=1}^{N} P_j \otimes \pi(\psi_j)$. This can be used to express $\pi(W_\Sigma)$.

More on test functions

Suppose $\Psi = \{\psi_1, \dots, \psi_d\}$ is a finite collection of test functions and P_js are orthogonal projections summing to the identity on $\mathcal{E}$.

A unital representation $\rho : C(\Psi) \to B(\mathcal{E})$ has the form $\rho(f) = \sum_{j=1}^N P_j\, f(\psi_j)$.

Set $Z(x) = \sum_{j=1}^N P_j \psi_j(x)$ — so $Z(x) = \rho(E(x))$, where $E(x) \in C(\Psi)$ is evaluation at x.

The Agler decomposition can be phrased as

$$[1] - \varphi\,\varphi^* = \Gamma \star ([1] - E^* E), \qquad \Gamma : X \times X \to C(\Psi)^* \text{ positive.}$$

A positive kernel Γ has a Kolmogorov decomposition $\Gamma(x, y) = \gamma^*(y)\gamma(x)$, where $\gamma : C(\Psi)^* \to \mathcal{E}$ for some Hilbert space $\mathcal{E}$, and a GNS-type construction gives a repersentation $\rho : C(\Psi) \to \mathcal{B}(\mathcal{H})$ with $\rho(a)\gamma(x)b = \gamma(x)ab$.

So the Agler decomposition becomes

$$[1] - \varphi\,\varphi^* = \gamma^* \star ([1] - Z^* Z) \star \gamma.$$

Given a representation π of $H^\infty(\mathcal{K}_\Psi)$ in $B(\mathcal{H})$, it is natural to define $\pi(Z) = \sum_{j=1}^N P_j \otimes \pi(\psi_j)$. This can be used to express $\pi(W_\Sigma)$.

The above works even with infinitely many test functions, though ρ isn't so explicit.

Theorem 3 (Dritschel & McCullough).

Suppose Ψ is a collection of test functions. The following are equivalent:

(MB) $\varphi \in H_1^\infty(\mathcal{K}_\Psi)$;

(AD) *There exists a positive kernel* $\Gamma : X \times X \to C_b(\Psi)$ *such that*

$$[1] - \varphi^*\varphi = \Gamma \star ([1] - EE^*);$$

(TF) *There is a unitary colligation* Σ *so that* $\varphi = W_\Sigma$;

(vNn) $\|\pi(f)\| \leq 1$ *for each continuous unital representation* π *of* $H^\infty(\mathcal{K}_\Psi)$ *which is strictly contractive on the test functions in* Ψ.

(vNw) $\|\pi(f)\| \leq 1$ *for each weakly continuous unital representation* π *of* $H^\infty(\mathcal{K}_\Psi)$ *which is contractive on the test functions in* Ψ.

Yet another realization theorem

Theorem 3 (Dritschel & McCullough).

Suppose Ψ is a collection of test functions. The following are equivalent:

(MB) $\varphi \in H^\infty_1(\mathcal{K}_\Psi)$;

(AD) *There exists a positive kernel $\Gamma : X \times X \to C_b(\Psi)$ such that*

$$[1] - \varphi^* \varphi = \Gamma \star ([1] - EE^*);$$

(TF) *There is a unitary colligation Σ so that $\varphi = W_\Sigma$;*

(vNn) $\|\pi(f)\| \leq 1$ *for each continuous unital representation π of $H^\infty(\mathcal{K}_\Psi)$ which is strictly contractive on the test functions in Ψ.*

(vNw) $\|\pi(f)\| \leq 1$ *for each weakly continuous unital representation π of $H^\infty(\mathcal{K}_\Psi)$ which is contractive on the test functions in Ψ.*

(vNs) $\|\pi(f)\| \leq 1$ *for each strongly continuous unital representation π of $H^\infty(\mathcal{K}_\Psi)$ which is contractive on the test functions in Ψ.*

Theorem 3 (Dritschel & McCullough).

Suppose Ψ *is a collection of test functions. The following are equivalent:*

(MB) $\varphi \in H_1^\infty(\mathcal{K}_\Psi)$;

(AD) *There exists a positive kernel* $\Gamma : X \times X \to C_b(\Psi)$ *such that*

$$[1] - \varphi^* \varphi = \Gamma \star ([1] - EE^*);$$

(TF) *There is a unitary colligation* Σ *so that* $\varphi = W_\Sigma$;

(vNn) $\|\pi(f)\| \leq 1$ *for each continuous unital representation* π *of* $H^\infty(\mathcal{K}_\Psi)$ *which is strictly contractive on the test functions in* Ψ.

(vNw) $\|\pi(f)\| \leq 1$ *for each weakly continuous unital representation* π *of* $H^\infty(\mathcal{K}_\Psi)$ *which is contractive on the test functions in* Ψ.

(vNs) $\|\pi(f)\| \leq 1$ *for each strongly continuous unital representation* π *of* $H^\infty(\mathcal{K}_\Psi)$ *which is contractive on the test functions in* Ψ.

A representation π is *weakly/strongly continuous* if whenever a bounded net (φ_α) converges pointwise to φ, $\pi(\varphi_\alpha)$ converges weakly/strongly to $\pi(\varphi)$.

Theorem 3 (Dritschel & McCullough).

Suppose Ψ is a collection of test functions. The following are equivalent:

(MB) $\varphi \in H_1^\infty(\mathcal{K}_\Psi)$;

(AD) *There exists a positive kernel $\Gamma : X \times X \to C_b(\Psi)$ such that*

$$[1] - \varphi^*\varphi = \Gamma \star ([1] - EE^*);$$

(TF) *There is a unitary colligation Σ so that $\varphi = W_\Sigma$;*

(vNn) $\|\pi(f)\| \leq 1$ *for each continuous unital representation π of $H^\infty(\mathcal{K}_\Psi)$ which is strictly contractive on the test functions in Ψ.*

(vNw) $\|\pi(f)\| \leq 1$ *for each weakly continuous unital representation π of $H^\infty(\mathcal{K}_\Psi)$ which is contractive on the test functions in Ψ.*

(vNs) $\|\pi(f)\| \leq 1$ *for each strongly continuous unital representation π of $H^\infty(\mathcal{K}_\Psi)$ which is contractive on the test functions in Ψ.*

A representation π is *weakly/strongly continuous* if whenever a bounded net (φ_α) converges pointwise to φ, $\pi(\varphi_\alpha)$ converges weakly/strongly to $\pi(\varphi)$. Strongly continuous is analogous to the $C_{0\cdot}$ condition.

Theorem 4 (Grinshpan, Kalyuzhni-Verbovetski, Vinnikov, Woerdeman).

Suppose $f \in H^\infty(\mathbb{D}^d)$. The following are equivalent.

(BM) $f \in H^\infty_1(\mathbb{D}^d)$;

(AD) *For any $p < q \in \{1, \dots, d\}$*

$$1 - f(z)\, f^*(w) = \Gamma_p(z,w) \prod_{j \neq p} (1 - Z_j(z) Z_j^*(w)) + \Gamma_q(z,w) \prod_{j \neq q} (1 - Z_j(z) Z_j^*(w)),$$

Γ_p, Γ_q *positive kernels;*

Agler decompositions for $H_1^\infty(\mathbb{D}^d)$

Theorem 4 (Grinshpan, Kalyuzhni-Verbovetski, Vinnikov, Woerdeman).

Suppose $f \in H^\infty(\mathbb{D}^d)$. The following are equivalent.

(BM) $f \in H_1^\infty(\mathbb{D}^d)$;

(AD) *For any $p < q \in \{1, \ldots, d\}$*

$$1 - f(z)\, f^*(w) = \Gamma_p(z,w) \prod_{j \neq p} (1 - Z_j(z) Z_j^*(w)) + \Gamma_q(z,w) \prod_{j \neq q} (1 - Z_j(z) Z_j^*(w)),$$

Γ_p, Γ_q *positive kernels;*

Greg Knese has a further refinement of this result.

Theorem 4 (Grinshpan, Kalyuzhni-Verbovetski, Vinnikov, Woerdeman).

Suppose $f \in H^\infty(\mathbb{D}^d)$. The following are equivalent.

(BM) $f \in H_1^\infty(\mathbb{D}^d)$;

(AD) *For any $p < q \in \{1, \ldots, d\}$*

$$1 - f(z)\, f^*(w) = \Gamma_p(z,w) \prod_{j \neq p} (1 - Z_j(z) Z_j^*(w)) + \Gamma_q(z,w) \prod_{j \neq q} (1 - Z_j(z) Z_j^*(w)),$$

Γ_p, Γ_q *positive kernels;*

Greg Knese has a further refinement of this result.

The Big Question redux

What about the rest of the realization theorem? Do we really need to initially assume $f \in H^\infty(\mathbb{D}^d)$?

Connections to real algebraic geometry?

Let $\{p_1, \ldots p_n\}$ be a collection of real polynomials on $\mathbb{R}^d$. The set $\mathcal{S} = \{x \in \mathbb{R}^d : p_k(x) \geq 0 \text{ for all } k\}$ is called a *(basic) semi-algebraic set*.

Connections to real algebraic geometry?

Let $\{p_1, \ldots p_n\}$ be a collection of real polynomials on $\mathbb{R}^d$. The set $\mathcal{S} = \{x \in \mathbb{R}^d : p_k(x) \geq 0$ for all $k\}$ is called a *(basic) semi-algebraic set*.

What sorts of polynomials are nonnegative on $\mathcal{S}$?

Connections to real algebraic geometry?

Let $\{p_1, \dots p_n\}$ be a collection of real polynomials on $\mathbb{R}^d$. The set $\mathcal{S} = \{x \in \mathbb{R}^d : p_k(x) \geq 0 \text{ for all } k\}$ is called a *(basic) semi-algebraic set*.

What sorts of polynomials are nonnegative on $\mathcal{S}$?

- Sums of squares: $\sum_j q_j^2$ (sum finite, q_js polynomials);

Connections to real algebraic geometry?

Let $\{p_1, \dots p_n\}$ be a collection of real polynomials on $\mathbb{R}^d$. The set $\mathcal{S} = \{x \in \mathbb{R}^d : p_k(x) \geq 0 \text{ for all } k\}$ is called a *(basic) semi-algebraic set*.

What sorts of polynomials are nonnegative on $\mathcal{S}$?

- Sums of squares: $\sum_j q_j^2$ (sum finite, q_js polynomials);
- Quadratic module: $\sum_k p_k \sum_j q_{kj}^2$;

Connections to real algebraic geometry?

Let $\{p_1, \ldots p_n\}$ be a collection of real polynomials on $\mathbb{R}^d$. The set $\mathcal{S} = \{x \in \mathbb{R}^d : p_k(x) \geq 0$ for all $k\}$ is called a *(basic) semi-algebraic set*.

What sorts of polynomials are nonnegative on $\mathcal{S}$?

- Sums of squares: $\sum_j q_j^2$ (sum finite, q_js polynomials);
- Quadratic module: $\sum_k p_k \sum_j q_{kj}^2$;
- Preordering: $\sum_\epsilon \prod p_k^{\epsilon_k} \sum_j q_{\epsilon j}^2$, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, each ϵ_k either 0 or 1.

Connections to real algebraic geometry?

Let $\{p_1, \dots p_n\}$ be a collection of real polynomials on $\mathbb{R}^d$. The set $\mathcal{S} = \{x \in \mathbb{R}^d : p_k(x) \geq 0 \text{ for all } k\}$ is called a *(basic) semi-algebraic set*.

What sorts of polynomials are nonnegative on $\mathcal{S}$?

- Sums of squares: $\sum_j q_j^2$ (sum finite, q_js polynomials);
- Quadratic module: $\sum_k p_k \sum_j q_{kj}^2$;
- Preordering: $\sum_\epsilon \prod p_k^{\epsilon_k} \sum_j q_{\epsilon j}^2$, $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, each ϵ_k either 0 or 1.

Theorem 5 (Schmüdgen's theorem).
Suppose f is a strictly positive polynomial on a compact semialgebraic set $\mathcal{S}$. Then f is in the preordering.

Connections to real algebraic geometry?

Let $\{p_1, \dots p_n\}$ be a collection of real polynomials on $\mathbb{R}^d$. The set $\mathcal{S} = \{x \in \mathbb{R}^d : p_k(x) \geq 0 \text{ for all } k\}$ is called a *(basic) semi-algebraic set*.

What sorts of polynomials are nonnegative on $\mathcal{S}$?

- Sums of squares: $\sum_j q_j^2$ (sum finite, q_js polynomials);
- Quadratic module: $\sum_k p_k \sum_j q_{kj}^2$;
- Preordering: $\sum_\epsilon \prod p_k^{\epsilon_k} \sum_j q_{\epsilon j}^2$, $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, each ϵ_k either 0 or 1.

Theorem 5 (Schmüdgen's theorem).
Suppose f is a strictly positive polynomial on a compact semialgebraic set $\mathcal{S}$. Then f is in the preordering.

The GK-VVW theorem is a sort of complex version of Schmüdgen's theorem.

Some notation

- Assume we have a finite set of test functions Ψ, $|\Psi| = d$, on a set X.

Some notation

- Assume we have a finite set of test functions Ψ, $|\Psi| = d$, on a set X.
- By a *preordering* Λ we will mean a finite subset of $\mathbb{N}^d$ with the partial ordering $n \leq m$ iff $n_i \leq m_i$, $i = 1, \ldots, d$. Write e_i for the tuple which is 1 at the ith entry and zero elsewhere. We require that Λ contain all e_is.

Some notation

- Assume we have a finite set of test functions Ψ, $|\Psi| = d$, on a set X.
- By a *preordering* Λ we will mean a finite subset of $\mathbb{N}^d$ with the partial ordering $n \leq m$ iff $n_i \leq m_i$, $i = 1, \ldots, d$. Write e_i for the tuple which is 1 at the ith entry and zero elsewhere. We require that Λ contain all e_is.
- It happens that in what follows there is no loss of generality in assuming if $\lambda \in \Lambda$ and $\lambda' \leq \lambda$, the $\lambda' \in \Lambda$.

Some notation

- Assume we have a finite set of test functions Ψ, $|\Psi| = d$, on a set X.
- By a *preordering* Λ we will mean a finite subset of $\mathbb{N}^d$ with the partial ordering $n \leq m$ iff $n_i \leq m_i$, $i = 1, \ldots, d$. Write e_i for the tuple which is 1 at the ith entry and zero elsewhere. We require that Λ contain all e_is.
- It happens that in what follows there is no loss of generality in assuming if $\lambda \in \Lambda$ and $\lambda' \leq \lambda$, the $\lambda' \in \Lambda$.
- An important case will be that of *ample preorderings*, which have the property that there is some $\lambda_m \in \Lambda$ such that for all $\lambda \in \Lambda$, $\lambda \leq \lambda_m$.

Some notation

- Assume we have a finite set of test functions Ψ, $|\Psi| = d$, on a set X.
- By a *preordering* Λ we will mean a finite subset of $\mathbb{N}^d$ with the partial ordering $n \leq m$ iff $n_i \leq m_i$, $i = 1, \ldots, d$. Write e_i for the tuple which is 1 at the ith entry and zero elsewhere. We require that Λ contain all e_is.
- It happens that in what follows there is no loss of generality in assuming if $\lambda \in \Lambda$ and $\lambda' \leq \lambda$, the $\lambda' \in \Lambda$.
- An important case will be that of *ample preorderings*, which have the property that there is some $\lambda_m \in \Lambda$ such that for all $\lambda \in \Lambda$, $\lambda \leq \lambda_m$.
- For $\lambda \in \Lambda$, define $\psi^\lambda := \psi_1^{\lambda_1} \cdots \psi_d^{\lambda_d}$.

Some notation

- Assume we have a finite set of test functions Ψ, $|\Psi| = d$, on a set X.
- By a *preordering* Λ we will mean a finite subset of $\mathbb{N}^d$ with the partial ordering $n \leq m$ iff $n_i \leq m_i$, $i = 1, \ldots, d$. Write e_i for the tuple which is 1 at the ith entry and zero elsewhere. We require that Λ contain all e_is.
- It happens that in what follows there is no loss of generality in assuming if $\lambda \in \Lambda$ and $\lambda' \leq \lambda$, the $\lambda' \in \Lambda$.
- An important case will be that of *ample preorderings*, which have the property that there is some $\lambda_m \in \Lambda$ such that for all $\lambda \in \Lambda$, $\lambda \leq \lambda_m$.
- For $\lambda \in \Lambda$, define $\psi^\lambda := \psi_1^{\lambda_1} \cdots \psi_d^{\lambda_d}$.
- The collection of kernels

$$\mathcal{K}_\Lambda := \Big\{ k : X \times X \to \mathbb{C} : k \geq 0 \text{ and for each } \lambda \in \Lambda,$$

$$\textstyle\prod_{\lambda \ni \lambda_i \neq 0} ([1] - \psi_i \psi_i^*)^{\lambda_i} * k \geq 0 \Big\},$$

are termed the *admissible kernels*. (Can be defined for $\mathcal{L}(\mathcal{H})$ as well).

Some notation

- Assume we have a finite set of test functions Ψ, $|\Psi| = d$, on a set X.
- By a *preordering* Λ we will mean a finite subset of $\mathbb{N}^d$ with the partial ordering $n \leq m$ iff $n_i \leq m_i$, $i = 1, \dots, d$. Write e_i for the tuple which is 1 at the ith entry and zero elsewhere. We require that Λ contain all e_is.
- It happens that in what follows there is no loss of generality in assuming if $\lambda \in \Lambda$ and $\lambda' \leq \lambda$, the $\lambda' \in \Lambda$.
- An important case will be that of *ample preorderings*, which have the property that there is some $\lambda_m \in \Lambda$ such that for all $\lambda \in \Lambda$, $\lambda \leq \lambda_m$.
- For $\lambda \in \Lambda$, define $\psi^\lambda := \psi_1^{\lambda_1} \cdots \psi_d^{\lambda_d}$.
- The collection of kernels

$$\mathcal{K}_\Lambda := \Big\{ k : X \times X \to \mathbb{C} : k \geq 0 \text{ and for each } \lambda \in \Lambda,$$

$$\textstyle\prod_{\lambda \ni \lambda_i \neq 0} ([1] - \psi_i \psi_i^*)^{\lambda_i} * k \geq 0 \Big\},$$

  are termed the *admissible kernels*. (Can be defined for $\mathcal{L}(\mathcal{H})$ as well).
- $H^\infty(\mathcal{K}_\Lambda)$ consisting of those functions φ on X for which there is a finite constant $C \geq 0$ such that for all $k \in \mathcal{K}_\Lambda$,

$$(C^2[1] - \varphi\varphi^*) \star k \geq 0,$$

  and $\|\varphi\|$ is defined to be the smallest such C.

Some notation

- Assume we have a finite set of test functions Ψ, $|\Psi| = d$, on a set X.
- By a *preordering* Λ we will mean a finite subset of $\mathbb{N}^d$ with the partial ordering $n \leq m$ iff $n_i \leq m_i$, $i = 1, \ldots, d$. Write e_i for the tuple which is 1 at the ith entry and zero elsewhere. We require that Λ contain all e_is.
- It happens that in what follows there is no loss of generality in assuming if $\lambda \in \Lambda$ and $\lambda' \leq \lambda$, the $\lambda' \in \Lambda$.
- An important case will be that of *ample preorderings*, which have the property that there is some $\lambda_m \in \Lambda$ such that for all $\lambda \in \Lambda$, $\lambda \leq \lambda_m$.
- For $\lambda \in \Lambda$, define $\psi^\lambda := \psi_1^{\lambda_1} \cdots \psi_d^{\lambda_d}$.
- The collection of kernels

$$\mathcal{K}_\Lambda := \Big\{ k : X \times X \to \mathbb{C} : k \geq 0 \text{ and for each } \lambda \in \Lambda,$$

$$\textstyle\prod_{\lambda \ni \lambda_i \neq 0} ([1] - \psi_i \psi_i^*)^{\lambda_i} * k \geq 0 \Big\},$$

  are termed the *admissible kernels*. (Can be defined for $\mathcal{L}(\mathcal{H})$ as well).
- $H^\infty(\mathcal{K}_\Lambda)$ consisting of those functions φ on X for which there is a finite constant $C \geq 0$ such that for all $k \in \mathcal{K}_\Lambda$,

$$(C^2[1] - \varphi\varphi^*) \star k \geq 0,$$

  and $\|\varphi\|$ is defined to be the smallest such C.
- The *generalized Schur-Agler class* is $H_1^\infty(\mathcal{K}_\Lambda)$, the unit ball of $H^\infty(\mathcal{K}_\Lambda)$.

More notation

- Define vectors of length 2^{d-1} from the components of any $\psi \in \Lambda$ by

$$\psi_\lambda^+(x)^* = \begin{pmatrix} \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_3}(x)^*\psi_{\lambda_4}(x)^* \\ \psi_{\lambda_{d-1}}(x)^*\psi_{\lambda_d}(x)^* \\ \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_3}(x)^* \\ \psi_{\lambda_2}(x)^*\psi_{\lambda_1}(x)^* \\ 1 \end{pmatrix} \text{ and } \psi_\lambda^-(x)^* = \begin{pmatrix} \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_4}(x)^* \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_3}(x)^* \\ \psi_{\lambda_d}(x)^* \\ \vdots \\ \psi_{\lambda_2}(x)^* \\ \psi_{\lambda_1}(x)^* \end{pmatrix}$$

More notation

- Define vectors of length 2^{d-1} from the components of any $\psi \in \Lambda$ by

$$\psi_\lambda^+(x)^* = \begin{pmatrix} \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_3}(x)^*\psi_{\lambda_4}(x)^* \\ \psi_{\lambda_{d-1}}(x)^*\psi_{\lambda_d}(x)^* \\ \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_3}(x)^* \\ \psi_{\lambda_2}(x)^*\psi_{\lambda_1}(x)^* \\ 1 \end{pmatrix} \text{ and } \psi_\lambda^-(x)^* = \begin{pmatrix} \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_4}(x)^* \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_3}(x)^* \\ \psi_{\lambda_d}(x)^* \\ \vdots \\ \psi_{\lambda_2}(x)^* \\ \psi_{\lambda_1}(x)^* \end{pmatrix}$$

- For $\lambda \in \Lambda$,

$$\prod_{\lambda_i \in \lambda}([1] - \psi_{\lambda_i}^* \psi_{\lambda_i})(x,y) = \psi_\lambda^+(x)\psi_\lambda^+(y)^* - \psi_\lambda^-(x)\psi_\lambda^-(y)^*$$

More notation

- Define vectors of length 2^{d-1} from the components of any $\psi \in \Lambda$ by

$$\psi_\lambda^+(x)^* = \begin{pmatrix} \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_3}(x)^*\psi_{\lambda_4}(x)^* \\ \psi_{\lambda_{d-1}}(x)^*\psi_{\lambda_d}(x)^* \\ \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_3}(x)^* \\ \psi_{\lambda_2}(x)^*\psi_{\lambda_1}(x)^* \\ 1 \end{pmatrix} \text{ and } \psi_\lambda^-(x)^* = \begin{pmatrix} \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_4}(x)^* \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_3}(x)^* \\ \psi_{\lambda_d}(x)^* \\ \vdots \\ \psi_{\lambda_2}(x)^* \\ \psi_{\lambda_1}(x)^* \end{pmatrix}$$

- For $\lambda \in \Lambda$,

$$\textstyle\prod_{\lambda_i \in \lambda}([1] - \psi_{\lambda_i}^*\psi_{\lambda_i})(x,y) = \psi_\lambda^+(x)\psi_\lambda^+(y)^* - \psi_\lambda^-(x)\psi_\lambda^-(y)^*$$

- By Douglas' lemma, for each $\lambda \in \Lambda$ there exists σ_λ such that $\psi_\lambda^-(x) = \psi_\lambda^+(x)\sigma_\lambda(x)$. In fact since $\psi_\lambda^+(x)$ is right invertible, we can set $\sigma_\lambda(x) = \psi_\lambda^+(x)^{-1}\psi_\lambda^-(x)$.

More notation

- Define vectors of length 2^{d-1} from the components of any $\psi \in \Lambda$ by

$$\psi_\lambda^+(x)^* = \begin{pmatrix} \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_3}(x)^*\psi_{\lambda_4}(x)^* \\ \psi_{\lambda_{d-1}}(x)^*\psi_{\lambda_d}(x)^* \\ \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_3}(x)^* \\ \psi_{\lambda_2}(x)^*\psi_{\lambda_1}(x)^* \\ 1 \end{pmatrix} \text{ and } \psi_\lambda^-(x)^* = \begin{pmatrix} \vdots \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_4}(x)^* \\ \psi_{\lambda_1}(x)^*\psi_{\lambda_2}(x)^*\psi_{\lambda_3}(x)^* \\ \psi_{\lambda_d}(x)^* \\ \vdots \\ \psi_{\lambda_2}(x)^* \\ \psi_{\lambda_1}(x)^* \end{pmatrix}$$

- For $\lambda \in \Lambda$,

$$\prod_{\lambda_i \in \lambda}([1] - \psi_{\lambda_i}^*\psi_{\lambda_i})(x,y) = \psi_\lambda^+(x)\psi_\lambda^+(y)^* - \psi_\lambda^-(x)\psi_\lambda^-(y)^*$$

- By Douglas' lemma, for each $\lambda \in \Lambda$ there exists σ_λ such that $\psi_\lambda^-(x) = \psi_\lambda^+(x)\sigma_\lambda(x)$. In fact since $\psi_\lambda^+(x)$ is right invertible, we can set $\sigma_\lambda(x) = \psi_\lambda^+(x)^{-1}\psi_\lambda^-(x)$.
- We refer to the functions σ_λ, $\lambda \in \Lambda$, as the *auxiliary test functions*.

Auxiliary test functions: the ample case

- An easy calculation shows

$$
\begin{aligned}
&\left(\psi_\lambda^+(x)(1_n - \sigma_\lambda(x)\sigma_\lambda(y)^*)k(x,y)\psi_\lambda^+(y)^*\right) \\
= &\left(\prod\nolimits_{\lambda_i \in \lambda}([1] - \psi_i\psi_i^*)^{\lambda_i} * k\right)(x,y) \geq 0.
\end{aligned}
$$

Auxiliary test functions: the ample case

- An easy calculation shows

$$\begin{aligned} &\left(\psi_\lambda^+(x)(1_n - \sigma_\lambda(x)\sigma_\lambda(y)^*)k(x,y)\psi_\lambda^+(y)^*\right) \\ = &\left(\prod_{\lambda_i \in \lambda}([1] - \psi_i\psi_i^*)^{\lambda_i} * k\right)(x,y) \geq 0. \end{aligned}$$

- However we do not necessarily have $([1] - \sigma_\lambda\sigma_\lambda^*) * k \geq 0$!

Auxiliary test functions: the ample case

- An easy calculation shows

$$\begin{aligned}&\left(\psi_\lambda^+(x)(1_n - \sigma_\lambda(x)\sigma_\lambda(y)^*)k(x,y)\psi_\lambda^+(y)^*\right)\\ =&\left(\prod_{\lambda_i\in\lambda}([1] - \psi_i\psi_i^*)^{\lambda_i} * k\right)(x,y) \geq 0.\end{aligned}$$

- However we do not necessarily have $([1] - \sigma_\lambda\sigma_\lambda^*) * k \geq 0$!
- This can be rectified in the ample case.

Auxiliary test functions: the ample case

- An easy calculation shows

$$\begin{aligned}&\left(\psi_\lambda^+(x)(1_n - \sigma_\lambda(x)\sigma_\lambda(y)^*)k(x,y)\psi_\lambda^+(y)^*\right)\\ =&\left(\prod_{\lambda_i\in\lambda}([1]-\psi_i\psi_i^*)^{\lambda_i} * k\right)(x,y) \geq 0.\end{aligned}$$

- However we do not necessarily have $([1]-\sigma_\lambda\sigma_\lambda^*) * k \geq 0$!
- This can be rectified in the ample case.

Lemma 6.
If Λ is ample, for each $\lambda \in \Lambda$ we can extend σ_λ to an $M(\mathbb{C}^{2^{|\lambda|-1}})$ valued function such that $([1]-\sigma_\lambda\sigma_\lambda^*) * k \geq 0$.

Auxiliary test functions: the ample case

- An easy calculation shows

$$\begin{aligned}&\left(\psi_\lambda^+(x)(1_n - \sigma_\lambda(x)\sigma_\lambda(y)^*)k(x,y)\psi_\lambda^+(y)^*\right)\\ = &\left(\prod_{\lambda_i\in\lambda}([1]-\psi_i\psi_i^*)^{\lambda_i} * k\right)(x,y) \geq 0.\end{aligned}$$

- However we do not necessarily have $([1]-\sigma_\lambda\sigma_\lambda^*) * k \geq 0$!
- This can be rectified in the ample case.

Lemma 6.
If Λ is ample, for each $\lambda \in \Lambda$ we can extend σ_λ to an $M(\mathbb{C}^{2^{|\lambda|-1}})$ valued function such that $([1]-\sigma_\lambda\sigma_\lambda^*) * k \geq 0$.

In particular, each auxiliary test function is in $H^\infty(\Lambda, M(\mathbb{C}^n))$ for an appropriate n.

Brehmer representations

Let π be a unital representation of $H^\infty(\mathcal{K}_\Lambda)$. We call π a *Brehmer representation* if for all $\lambda \in \Lambda$,

$$\prod_{\lambda \ni \lambda_i \neq 0} (1 - \pi(\psi_i)\pi(\psi_i)^*)^{\lambda_i} \geq 0.$$

Let π be a unital representation of $H^\infty(\mathcal{K}_\Lambda)$. We call π a *Brehmer representation* if for all $\lambda \in \Lambda$,

$$\prod_{\lambda \ni \lambda_i \neq 0} (1 - \pi(\psi_i)\pi(\psi_i)^*)^{\lambda_i} \geq 0.$$

A representation π of $H^\infty(\mathcal{K}_\Lambda)$ is a *strict Brehmer representation* if the inequalities are strict. It is a *norm/strongly/weakly continuous Brehmer representation* if it is a Brehmer representation and whenever a bounded net (φ_α) converges pointwise in norm to φ (in other words, $\sup_\alpha \|\varphi_\alpha\|_\infty < \infty$ and for each $x \in X$, $\|\varphi_\alpha(x) - \varphi(x)\| \to 0$), $\pi(\varphi_\alpha)$ converges in norm/strongly/weakly to $\pi(\varphi)$.

Theorem 7 (Realization theorem I).

The following are equivalent:

1. $\varphi \in H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$ *and* $\|\varphi\| \le 1$.
2. φ *has an Agler decomposition; that is, there is a completely positive kernel* $\Gamma : X \times X \to \mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ *such that for all* $x, y \in X$,

$$1 - \varphi(x)\varphi(y)^* = \sum_\lambda \Gamma_\lambda(x,y)\left(\psi_\lambda^+(x)\psi_\lambda^+(y)^* - \psi_\lambda^-(x)\psi_\lambda^-(y)^*\right) \otimes 1_{\mathcal{L}(\mathcal{H})},$$

for some positive kernels Γ_λ.

Furthermore, in this situation, φ *has a transfer function representation: there exists a unitary colligation* Σ *such that*

$$\varphi = W_\Sigma(x) := D + CS(x)(1 - AS(x))^{-1}B,$$

where $S(x) = \sum_\lambda \sigma_\lambda \otimes P_\lambda$.

Theorem 7 (Realization theorem I).

The following are equivalent:

1. $\varphi \in H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$ *and* $\|\varphi\| \leq 1$.
2. φ *has an Agler decomposition; that is, there is a completely positive kernel* $\Gamma : X \times X \to \mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ *such that for all* $x, y \in X$,

$$1 - \varphi(x)\varphi(y)^* = \sum_\lambda \Gamma_\lambda(x,y) \left(\psi_\lambda^+(x)\psi_\lambda^+(y)^* - \psi_\lambda^-(x)\psi_\lambda^-(y)^*\right) \otimes 1_{\mathcal{L}(\mathcal{H})},$$

for some positive kernels Γ_λ.

Furthermore, in this situation, φ *has a transfer function representation: there exists a unitary colligation* Σ *such that*

$$\varphi = W_\Sigma(x) := D + CS(x)(1 - AS(x))^{-1}B,$$

where $S(x) = \sum_\lambda \sigma_\lambda \otimes P_\lambda$.

Note the subtle difference between this and the usual realisation theorem: Given a unitary colligation Σ, there is no guarantee that $W_\Sigma \in H^\infty(\mathcal{K}_\Lambda)$.

Another realization theorem

Theorem 7 (Realization theorem I).

The following are equivalent:

1. $\varphi \in H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$ *and* $\|\varphi\| \leq 1$.
2. φ *has an Agler decomposition; that is, there is a completely positive kernel* $\Gamma : X \times X \to \mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ *such that for all* $x, y \in X$,

$$1 - \varphi(x)\varphi(y)^* = \sum_\lambda \Gamma_\lambda(x,y)\left(\psi_\lambda^+(x)\psi_\lambda^+(y)^* - \psi_\lambda^-(x)\psi_\lambda^-(y)^*\right) \otimes 1_{\mathcal{L}(\mathcal{H})},$$

for some positive kernels Γ_λ.

Furthermore, in this situation, φ *has a transfer function representation: there exists a unitary colligation* Σ *such that*

$$\varphi = W_\Sigma(x) := D + CS(x)(1 - AS(x))^{-1}B,$$

where $S(x) = \sum_\lambda \sigma_\lambda \otimes P_\lambda$.

Note the subtle difference between this and the usual realisation theorem: Given a unitary colligation Σ, there is no guarantee that $W_\Sigma \in H^\infty(\mathcal{K}_\Lambda)$.

This makes it difficult to do interpolation!

Theorem 8 (Realization theorem II).

Let Λ be an ample preordering. The following are equivalent:

1. $\varphi \in H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$ *and* $\|\varphi\| \leq 1$.
2. φ *has an Agler decomposition; that is, there is a completely positive kernel* $\Gamma : X \times X \to \mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ *such that for all* $x, y \in X$,

$$1 - \varphi(x)\varphi(y)^* = \sum_\lambda \Gamma_\lambda(x,y)\left(\psi_\lambda^+(x)\psi_\lambda^+(y)^* - \psi_\lambda^-(x)\psi_\lambda^-(y)^*\right) \otimes 1_{\mathcal{L}(\mathcal{H})},$$

   for some positive kernels Γ_λ.
3. φ *has a transfer function representation*

$$\varphi = W_\Sigma(x) := D + CS(x)(1 - AS(x))^{-1}B,$$

Furthermore, in this case every weakly continuous Brehmer representation is completely contractive.

Theorem 9 (Realization theorem III).

Let $X = \mathbb{D}^d$, $\Psi = \{z_j\}$, $\Lambda = \{\lambda \leq (1, \dots, 1)\}$. *The following are equivalent:*

1. $\varphi \in H^\infty(\mathbb{D}^d)$ *and* $\|\varphi\|_\infty \leq 1$.
2. φ *has an Agler decomposition; that is, there is a completely positive kernel* $\Gamma : X \times X \to \mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ *such that for all* $x, y \in X$,

$$1 - \varphi(x)\varphi(y)^* = \sum_\lambda \Gamma_\lambda(x, y) \left(\psi_\lambda^+(x)\psi_\lambda^+(y)^* - \psi_\lambda^-(x)\psi_\lambda^-(y)^*\right) \otimes 1_{\mathcal{L}(\mathcal{H})},$$

   for some positive kernels Γ_λ.
3. φ *has a transfer function representation*

$$\varphi = W_\Sigma(x) := D + CS(x)(1 - AS(x))^{-1}B,$$

Furthermore, in this case every weakly continuous Brehmer representation is completely contractive.

The polydisk continued

Corollary 10.

Any weakly continuous 2^{d-1}*-contractive representation of* $H^\infty(\mathbb{D}^d)$ *is completely contractive.*

Corollary 10.

Any weakly continuous 2^{d-1}-contractive representation of $H^\infty(\mathbb{D}^d)$ is completely contractive.

With a bit more work, we can replace 2^{d-1} by 2^{d-2}. In particular then, any strictly 2-contractive representation of $H^\infty(\mathbb{D}^3)$ is completely contractive.

Theorem 11 (Realization theorem II$'$).
Let $\Psi = \{\psi_1, \psi_2\}$ on $\mathbb{C}^d$, $\Lambda = \{\lambda \leq (1,1)\}$. The following are equivalent:

1. $\varphi \in H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$ *and* $\|\varphi\| \leq 1$.
2. φ *has an Agler decomposition; that is, there is a completely positive kernel* $\Gamma : X \times X \to \mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ *such that for all* $x, y \in X$,

$$1 - \varphi(x)\varphi(y)^* = \sum_\lambda \Gamma_i(x,y)\,(1 - \psi_i(x)\psi_i(y)^*) \otimes 1_{\mathcal{L}(\mathcal{H})},$$

   for some positive kernels Γ_λ.
3. φ *has a transfer function representation*

$$\varphi = W_\Sigma(x) := D + CS(x)(1 - AS(x))^{-1}B,$$

   where now $S(x) = \psi_1 P_1 \oplus \psi_2 P_2$.

Furthermore, in this case any weakly continuous representation π for which $\|\pi(\psi_i)\| \leq 1$, $i = 1, 2$, is completely contractive.