Toeplitz and Hankel operators on weighted Fock spaces

MIROSLAV ENGLIŠ (Prague)

ABSTRACT. We give criteria for the membership of Toeplitz operators and of products of Hankel operators, with symbols of a certain type, in Schatten ideals and in the Dixmier class, and formulas for their Dixmier trace, on a variety of weighted Segal-Bargmann-Fock spaces on the complex plane.

`joint with:` H. Bommier, E.H. Youssfi (Marseille)

Hardy space operators

$\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$

Hardy space: $L^2(\mathbf{T}) \supset H^2 = \{f : \widehat{f}(n) = 0 \; \forall n < 0\} = L^2_{\mathrm{hol}}(\mathbf{T})$,

$$\widehat{f}(n) := \frac{1}{2\pi} \int_0^1 f(e^{i\theta}) \, e^{-ni\theta} \, d\theta.$$

Szegö projection: $S : L^2(\mathbf{T}) \to H^2$.

Toeplitz operator with symbol $f \in L^\infty(\mathbf{T})$:

$$T_f^{\mathrm{Hardy}} : H^2 \to H^2, \quad g \mapsto S(fg)$$

Hankel operator:

$$H_f^{\mathrm{Hardy}} : H^2 \to L^2 \ominus H^2 = \overline{H_0^2}, \quad g \mapsto (I - S)(fg).$$

Bergman space operators

$\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$

Bergman space: $L^2(\mathbf{D}) \supset A^2 = \{f : f \text{ holomorphic on } \mathbf{D}\}$

Bergman projection: $P : L^2(\mathbf{D}) \to A^2$

Toeplitz operator with symbol $f \in L^\infty(\mathbf{D})$:

$$T_f : A^2 \to A^2, \quad g \mapsto P(fg);$$

Hankel operator:

$$H_f : A^2 \to L^2(\mathbf{D}) \ominus A^2, \quad g \mapsto (I - P)(fg).$$

Similarly — on the ball $\mathbf{B}^n \subset \mathbf{C}^n$, or any $\Omega \subset \mathbf{C}^n$:

Bergman space (wrt the Lebesgue measure)

Hardy space (Poisson extension holomorphic; wrt surface measure)

[bigger orthocomplement]

Szegö/Bergman projection, Toeplitz ops., Hankel ops.

H_f in Schatten classes

Hankel operators on $H^2(\mathbf{T})$ [Peller 1982] [Semmes 1984]:

for f holomorphic, $H_{\overline{f}} \in \mathcal{S}^p \iff f \in B^p$, the p-th Besov space:

$$\int_{\mathbf{D}} |f^{(k)}(z)|^p \, (1-|z|^2)^{kp-2} \, dm(z) < \infty,$$

for some ($\Leftrightarrow$ any) $k > 1/p$. $\qquad (0 < p < \infty)$ $\qquad$ (Trivial for $f \in C^\infty(\mathbf{T})$.)

Hankel operators on $A^2(\mathbf{D})$ [Arazy-Fisher-Janson-Peetre 1988]:

for f holomorphic, $\quad H_{\overline{f}} \in \mathcal{S}^p$, $1 < p < \infty$, $\iff f \in B^p$;

$$H_{\overline{f}} \in \mathcal{S}^p,\ p \le 1, \iff H_{\overline{f}} = 0.$$

... **Cut-off** at $p = 1$. $\qquad$ (Never $\mathcal{S}^1$ even for $f \in C^\infty(\overline{\mathbf{D}})$.)

Hankel operators on $A^2(\mathbf{B}^n)$, $n > 1$ [Arazy-Fisher-Peetre 1988]:

for f holomorphic, $$H_{\overline{f}} \in \mathcal{S}^p,\ 2n < p < \infty,\ \iff\ f \in B^p;$$
$$H_{\overline{f}} \in \mathcal{S}^p,\ p \leq 2n,\ \iff\ H_{\overline{f}} = 0.$$
(cut-off at $p = 2n$)

Same result — for any Ω smoothly bounded strictly pseudoconvex in $\mathbf{C}^n$, $n \geq 2$. [Li-Luecking 1995]

At the cutoff: *Dixmier ideal and Dixmier trace.*

The Dixmier ideal

Recall: a compact operator T on a Hilbert space belongs to $\mathcal{S}^p$ iff

$$\sum_j s_j(T)^p < \infty,$$

where $s_0(T) \geq s_1(T) \geq s_2(T) \geq \dots$ are the eigenvalues of $(T^*T)^{1/2}$ (counting multiplicities).

Schatten ideals: $T \in \mathcal{S}^p \iff \{s_j(T)\} \in \ell^p \qquad (0 < p < \infty);$

$$T \in \mathcal{S}^{p,\infty} \iff s_j(T) = O(j^{-1/p}) \qquad (0 < p < \infty).$$

($\mathcal{S}^1$ — trace class).

Dixmier ideal $\mathcal{S}^{\mathrm{Dixm}}$: consists of all T such that

$$\sum_{j=1}^{N} s_j(T) = O(\log N).$$

Norm:

$$\|T\|_{\mathrm{Dixm}} := \sup_{N \geq 2} \frac{1}{\log N} \sum_{j=0}^{N} s_j(T).$$

Inclusions: $\qquad \mathcal{S}^1 \subset \mathcal{S}^{1,\infty} \subset \mathcal{S}^{\mathrm{Dixm}} \subset \mathcal{S}^p, \qquad \forall p > 1.$

Dixmier trace

$\omega : l^\infty \to \mathbf{C}$ a Banach limit

i.e. $\omega \in (l^\infty)^*$ of norm 1 extending $\lim \in c^*$ on $c \subset l^\infty$

ω is scaling invariant if $\omega(x_1, x_1, x_2, x_2, \dots) = \omega(x_1, x_2, \dots)$

Dixmier trace of $A \in \mathcal{S}^{\mathrm{Dixm}}$, $A \geq 0$:

$$\mathrm{Tr}_\omega(A) := \omega\Big(\frac{1}{1 + \log n} \sum_{j=1}^{n} s_j(A)\Big).$$

One has $\mathrm{Tr}_\omega(A + B) = \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B)$ if ω scaling invariant; hence, makes sense to set

$$\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(A_+) - \mathrm{Tr}_\omega(A_-) \qquad \text{for } A = A_+ - A_- = A^* \in \mathcal{S}^{\mathrm{Dixm}},$$

$$\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega\left(\frac{A + A^*}{2}\right) + i\,\mathrm{Tr}_\omega\left(\frac{A - A^*}{2i}\right) \qquad \text{for arbitrary } A \in \mathcal{S}^{\mathrm{Dixm}}.$$

A is measurable if $\mathrm{Tr}_\omega(A)$ is the same for all ω.

Properties of Tr_ω:

- Cyclicity: $\mathrm{Tr}_\omega(AB) = \mathrm{Tr}_\omega(BA)$ $\quad A \in \mathcal{S}^{\mathrm{Dixm}}$, B bounded.

- Nontrivial: $\mathrm{Tr}_\omega(A) = 0$ $\quad$ if A trace-class.

- for $T \geq 0$

$$\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{j=0}^{N} s_j(T)$$

  if the limit exists.

[Connes – NG book 1994]

We had the cut-off phenomena: for f holomorphic on the disc,

$$H_{\overline{f}} \in \mathcal{S}^p \iff \begin{cases} f \in B^p, & 1 < p < \infty, \\ f = \text{const.}, & 0 < p \leq 1, \end{cases}$$

and f holomorphic on the ball or on s-pscvx $\Omega \subset \mathbf{C}^n$ s-pscvx, $n \geq 2$,

$$H_{\overline{f}} \in \mathcal{S}^p \iff \begin{cases} f \in B^p, & 2n < p < \infty, \\ f = \text{const.}, & 0 < p \leq 2n. \end{cases}$$

<u>**Question:**</u> When is

$$\begin{aligned} H_f &\in \mathcal{S}^{\text{Dixm}} \qquad \text{on} \quad \mathbf{D}, \\ H^*_{f_1} H_{f_2} \dots H^*_{f_{2n-1}} H_{f_{2n}} &\in \mathcal{S}^{\text{Dixm}} \qquad \text{on} \quad \mathbf{B}^n,\ n \geq 2, \end{aligned}$$

or on $\Omega \subset \mathbf{C}^n$, $n \geq 2$, nice?

What are the corresponding Dixmier traces

$$\operatorname{Tr}_\omega(|H_f|), \qquad \operatorname{Tr}_\omega(H^*_{f_1} H_{f_2} \dots H^*_{f_{2n-1}} H_{f_{2n}}) \ ?$$

<u>Known previously</u>:

- for $\mathbf{D}$ [Arazy-Fisher-Peetre 1988]

$$f \in B^1 \implies H_{\overline{f}} \in \mathcal{S}^{\mathrm{Dixm}},$$

  but not conversely.

- $0 \neq H_f \in \mathcal{I},$ a unitary ideal $\implies \mathcal{S}^{\mathrm{Dixm}} \subset \mathcal{I}.$
- Nothing known about $\mathrm{Tr}_\omega(|H_f|)$.

- Nothing known for $\mathbf{B}^n$, $n \geq 2$, or strictly-pseudoconvex.

Results — disc

Theorem. [R. Rochberg, ME] *If* $f \in C^\infty(\overline{\mathbf{D}})$, *then*

$$\mathrm{Tr}_\omega(|H_f|) = \int_{\mathbf{T}} |\overline{\partial} f| \, d\sigma,$$

where $d\sigma$ = *normalized arc-length.*

In particular: H_f measurable. (Nontrivial even for smooth f.)

Theorem. [RR, ME] *For* f *holomorphic on* $\mathbf{D}$, *TFAE:*

(1) $f' \in H^1$;

(2) $H_{\overline{f}} \in \mathcal{S}^{1,\infty}$;

(3) $H_{\overline{f}} \in \mathcal{S}^{\mathrm{Dixm}}$.

In that case $|H_{\overline{f}}|$ *is measurable and*

$$\mathrm{Tr}_\omega(|H_{\overline{f}}|) = \int_{\mathbf{T}} |f'| \, d\sigma = \|f'\|_{H^1}.$$

Theorem. $\Omega \subset \mathbf{C}^n$ *smoothly bounded strictly pseudoconvex. Then for any* $2n$ *functions* $f_1, g_1, \dots, f_n, g_n \in C^\infty(\overline{\Omega})$,

$$H^*_{f_1} H_{g_1} \dots H^*_{f_n} H_{g_n} =: H \in \mathcal{S}^{\mathrm{Dixm}}$$

and

$$\mathrm{Tr}_\omega(H) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \prod_{j=1}^n \mathcal{L}(\overline{\partial}_b g_j, \overline{\partial}_b f_j)\, d\mu,$$

where

$$d\mu := \tfrac{1}{2i^n}\, (\partial r - \overline{\partial} r) \wedge (\overline{\partial}\partial r)^{n-1}.$$

Here $\mathcal{L}$ *is the dual Levi form on* $\mathcal{T}''^*$ *(the dual of the anti-holomorphic complex tangent space on* $\partial\Omega$*), and* r *is a defining function for* Ω*.*

In particular, H is measurable.

Results — the Fock space

Fock (Segal-Bargmann) space:

$$\mathcal{F}_\gamma := \{f \in L^2(\mathbf{C}^n, e^{-\gamma|z|^2}(\tfrac{\gamma}{\pi})^n\,dz) : f \text{ entire}\}, \qquad \gamma > 0. \qquad (\gamma = \tfrac{1}{2})$$

Toeplitz and Hankel operators:

$$T_f : u \mapsto P(fu), \qquad H_f : u \mapsto (I - P)(fu),$$

where $f \in L^\infty(\mathbf{C}^n)$ and $P : L^2 \to \mathcal{F}_\gamma$ is the OG projection.

Analogue of C^∞ on the closure: $f \in \mathcal{A} \overset{\text{def}}{\Longleftrightarrow}$

$$f(z) \approx \sum_{j=0}^{\infty} f_j(z) \qquad \text{as } |z| \to +\infty,$$

where $f_j(z)$ is homogeneous of degree $-j$, i.e. $f_j(tz) = t^{-j} f_j(z)\ \forall t > 0$.
(Symbol classes for psdo's.)

Theorem. *For $f, g \in \mathcal{A}$ and $\zeta \in \mathbf{S}^{2n-1}$, denote*

$$Q(f,g) := \lim_{r\to+\infty} r^2 \sum_{j=1}^{n} \partial_j \overline{f}(r\zeta) \cdot \overline{\partial}_j g(r\zeta)$$

(*the limits exists thanks to the definition of $\mathcal{A}$, and in fact we may replace f, g by their top-degree components f_0, g_0*).

Then for any $f_1, g_1, \dots, f_n, g_n \in \mathcal{A}$, the product

$$H^*_{f_1} H_{g_1} \dots H^*_{f_n} H_{g_n} =: H$$

belongs to $\mathcal{S}^{\mathrm{Dixm}}$, is measurable, and

$$\mathrm{Tr}_\omega(H) = \frac{1}{n!} \int_{\mathbf{S}^{2n-1}} Q(f_1, g_1) \dots Q(f_n, g_n)\, d\sigma$$

where $d\sigma$ is the normalized surface measure on $\mathbf{S}^{2n-1}$.

Proof. Weyl operator with symbol $a = a(x, \xi)$ on $L^2(\mathbf{R}^n)$:

$$W_a f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} a\left(\frac{x+y}{2}, \xi\right) e^{-i\langle x-y, \xi\rangle} f(y)\, dy\, d\xi.$$

Converges if $a \in \mathcal{S}(\mathbf{R}^{2n})$; extends in standard way to $a \in \mathcal{S}'(\mathbf{R}^{2n})$.

Special cases: $W_{a(x)} = M_a$; $W_{\xi^\alpha} = i^\alpha \partial^\alpha$.

<u>Bargmann transform:</u> isometry $\beta : L^2(\mathbf{R}^n) \to \mathcal{F}_{-1/2}$,

$$\beta f(z) := \frac{1}{(4\pi^3)^{n/4}} \int_{\mathbf{R}^n} f(x) e^{x\cdot z - x\cdot x/2 - z\cdot z/4}\, dx.$$

(Extends also to general F_γ via dilations $z \mapsto \sqrt{\gamma} z$.)

<u>Fact:</u> Relationship to Toeplitz operators:

$$\beta^* T_a \beta = W_{\mathcal{E}a}$$

where $a(x, \xi)$ is also viewed as the function $a(z)$ of $z = x + i\xi \in \mathbf{C}^n$, and

$$\mathcal{E}a(z) = \left(\frac{2\gamma}{\pi}\right)^n \int_{\mathbf{C}^n} a(w) e^{-2\gamma|z-w|^2}\, dw = e^{\Delta/8\gamma} a(z)$$

is the heat solution at time $t = \frac{1}{8\gamma}$.

Reduces the problem to deciding when $W_{\mathcal{E}(fg)} - W_{\mathcal{E}f} W_{\mathcal{E}g} \in \mathcal{S}^{2n,\infty}$.

Using the interplay between W_a and $T_a = W_{\mathcal{E}a} = W_{a+\mathrm{LOT}}$:

Theorem.

(a) *Let $p > 1$ and $a \in \mathcal{A}^m$, $m < -2n/p$. Then $W_a, T_a \in \mathcal{S}^p$.*
(b) *Let $p > 1$ and $a \in \mathcal{A}^m$, $m \leq -2n/p$. Then $W_a, T_a \in \mathcal{S}^{p,\infty}$.*
(c) *If $a \in \mathcal{A}^0$, then W_a is bounded.*

(The last usually proved using Calderon-Vaillancourt — we are able to prove it using only that T_f is bounded for f bounded.)

Known for W_a; folk lore for Toeplitz.

Here $\mathcal{A}^m$ denotes the class of f with homogeneous expansion

$$f(z) \approx |z|^m \sum_{j=0}^{\infty} f_j(z) \qquad \text{as } |z| \to +\infty.$$

Rest of this talk: Generalizations to weighted Fock spaces.

Notation:

$$\mathcal{F}_w := \{f \in L^2(\mathbf{C}^n, w) : f \text{ is holomorphic}\}.$$

Here w is (positive continuous) assumed to be such that

$$|z|^k w(z) \text{ is integrable for all } k \geq 0,$$

so that polynomials belong to $\mathcal{F}_w$. In all cases we will consider here, they will also be dense in $\mathcal{F}_w$.

Example:

$$\mathcal{F}_m := \mathcal{F}_w \qquad \text{for} \qquad w(z) = e^{-|z|^{2m}},$$

“higher-order Fock” spaces. $\square$

Toeplitz and Hankel operators on $\mathcal{F}_w$:

$$T_f = P_+ M_f, \qquad H_f = P_- M_f,$$

where $P_+ : L^2(\mathbf{C}^n, w) \to \mathcal{F}_w$ is the orthogonal projection, $P_- = I - P_+$, and $M_f : u \mapsto fu$ is the operator of “multiplication by f”.

Question: Membership in $\mathcal{S}^p$, $\mathcal{S}^{\mathrm{Dixm}}$, Tr_ω.

Will work on $\mathbf{C}$; the case of $\mathbf{C}^n$ involves just more technicalities.

Related work:

- [Holland, Rochberg 2001] — radial weights, estimates for Bergman kernel, Hankel forms
- [Bommier-Hato, Youssfi 2007] — $H_{\overline{f}}$, f holomorphic, on $\mathcal{F}_m$
- [Seip, Youssfi 2012] — $\mathcal{S}^p$ criteria for $H_{\overline{f}}$, f holomorphic, similar (finer) estimates for Bergman kernel
- [Lin, Rochberg 1995], [Constantin, Ortega-Cerda 2011] (for $\overline{\partial}$), ...

Nothing known for $\mathcal{S}^{\text{Dixm}}$, Tr_ω.

Results

Theorem A. *Let*

$$w(z) = e^{-|z|^{2m}}, \qquad m > 0.$$

Assume that $f, g \in L^\infty(\mathbf{C})$ have the form

$$f(z) = \sum_{j=0}^{q} |z|^{-j} f_j\Big(\frac{z}{|z|}\Big) + O\Big(\frac{1}{|z|^{q+1}}\Big) \qquad \text{as } |z| \to +\infty$$

and similarly for g, where $q+1 > m$ and f_j, g_j, $j = 0, 1, \ldots, q$, are some functions in $C^\infty(\mathbf{T})$.
Extend f_0 to $\mathbf{C} \setminus \{0\}$ by $f_0(z) := f_0\Big(\frac{z}{|z|}\Big)$ and similarly for g_0.

Then $H_{\overline{f}}$, H_g belong to $\mathcal{S}^{2,\infty}$, the operator $H_{\overline{f}}^* H_g$ belongs to the Dixmier class, is measurable, the limit

$$Q(f,g)(e^{i\theta}) := \lim_{r\to+\infty} r^2 \partial f_0(re^{i\theta}) \overline{\partial} g_0(re^{i\theta})$$

exists for any $e^{i\theta} \in \mathbf{T}$, and

$$\operatorname{Tr}_\omega(H_{\overline{f}}^* H_g) = \frac{1}{2\pi m} \int_0^{2\pi} Q(f,g)(e^{i\theta})\, d\theta.$$

Also an analogous result for T_f.

The hypothesis $q+1>m$ is essential: for $f(z)=e^{i|z|^2}$, H_f is not even compact.

Theorem B. *Let*

$$w(z) = \rho(z)|z|^q e^{-|z|^2}$$

where $q \in \mathbf{R}$ *and*

$$\rho(z) \qquad \textit{is bounded and bounded away from 0 as } z \to \infty.$$

Then

(i) $f \in \mathcal{A}^s$, $s < 0 \implies T_f \in \mathcal{S}^{-2/s,\infty}$.
(ii) $f \in \mathcal{A}^s$, $s < -2/p \implies T_f \in \mathcal{S}^p$ $(p \geq 1)$.
(iii) $f, g \in \mathcal{A}^0 \implies H_f^* H_g \in \mathcal{S}^{\mathrm{Dixm}}$ *and* Tr_ω *is given by the same formula as for* $w(z) = e^{-|z|^2}$.

Proof of Theorem A — elementary methods.

Proof of Theorem B — reduction to Weyl calculus.

Fact C. *Let*

$$w(z) = e^{-|z|^{2m}}, \qquad m > 0.$$

Then there exists a natural "Bargman transform"

$$\beta_m : L^2(\mathbf{R}) \to \mathcal{F}_m$$

such that

$$\beta_m^* T_f \beta_m = W_{\mathcal{E}_m f}$$

where $\mathcal{E}_m$ *is a certain* ΨDO *on* $\mathbf{C}$.

We originally hoped to use this for giving a proof of Theorem A also by reduction to the Weyl calculus.
However, $\mathcal{E}_m$ is too complicated.

Proof of Theorem A

For convenience, introduce the notation ($s > 0$)

$$\mathcal{I}^{-s} = \mathcal{S}^{1/s,\infty} = \{T : s_j(T) = O(j^{-s}) \text{ as } j \to \infty\}.$$

Then $\mathcal{I}^{-s}$ is a vector space, with quasi-norm

$$\|T\|_{-s} := \sup_j (j+1)^s s_j(T)$$

satisfying

$$\|A + B\|_{-s} \leq 2^s(\|A\|_{-s} + \|B\|_{-s}),$$
$$\|AB\|_{-s-t} \leq 2^{s+t}\|A\|_{-s}\|B\|_{-t}.$$

In particular, $\mathcal{I}^{-s} \cdot \mathcal{I}^{-t} \subset \mathcal{I}^{-s-t}$.

Let also χ denote the characteristic function of the extrior disc $|z| \geq 1$.

Then the functions

$$|z|^s \chi(z)$$

are defined for any $s \in \mathbf{R}$ (we avoid the singularity $z = 0$ for negative s).

Observation 1: $T_{|z|^s\chi} \in \mathcal{I}^{s/2m}$.

Proof. For any radial function w on $\mathbf{C}$, denote by

$$c_k(w) := \int_{\mathbf{C}} |z|^{2k} w(z)\, dz$$

its moments. Then if ϕ is another radial function, the Toeplitz operator T_ϕ on $\mathcal{F}_w$ is diagonalized by the monomial basis $\{z^k\}$:

$$T_\phi :\; z^k \longmapsto \frac{c_k(\phi w)}{c_k(w)}\, z^k.$$

Applying this to $w(z) = e^{-|z|^{2m}}$, $\phi(z) = |z|^s\chi(z)$ gives the result, since

$$c_k(w) = \frac{\pi}{m}\Gamma\Big(\frac{m+1}{\pi}\Big), \qquad c_k(\phi w) \sim c_{k+s/2}(w),$$

and

$$\frac{c_{k+s/2}(w)}{c_k(w)} \sim \frac{\Gamma(\frac{k+\frac{s}{2}+1}{m})}{\Gamma(\frac{k+1}{m})} \sim k^{s/2m}$$

by Stirling's formula. $\square$

Observation 2: $T_f \in \mathcal{I}^{s/2m}$ if $f(z) = O(|z|^s)$ as $z \to \infty$.

Proof. Write
$$T_f = T_{(1-\chi)f} + T_{|z|^s \chi g}$$
with g bounded. Then

$$T_{|z|^s\chi g} = P_+ M_{|z|^s \chi g} P_+ = (M_{|z|^{s/2}\chi} P_+)^* M_g (M_{|z|^{s/2}\chi} P_+)$$

while
$$(M_{|z|^{s/2}\chi} P_+)^* (M_{|z|^{s/2}\chi} P_+) = T_{|z|^s \chi}.$$

By previous observation, the last product belongs to $\mathcal{I}^{s/2m}$, hence $(M_{|z|^{s/2}\chi} P_+) \in \mathcal{I}^{s/4m}$ and $T_{|z|^s\chi g} \in \mathcal{I}^{s/2m}$.

A similar argument shows that $T_{(1-\chi)f}$ in fact belongs to all $\mathcal{I}^s$, $s < 0$. So $T_f \in \mathcal{I}^{s/2m}$. $\square$

Using the formula

$$T_{fg} - T_f T_g = H^*_{\overline{f}} H_g,$$

similar argument yields also the following two observations.

Observation 3: $H_f \in \mathcal{I}^{s/2m}$ if $f(z) = O(|z|^s)$ as $z \to \infty$.

Observation 4: For $f \in L^\infty(\mathbf{C})$ and $s \leq 0$,

$$H_{|z|^s\chi},\ \ T_{f|z|^s\chi} - T_f T_{|z|^s\chi},\ \ T_{f|z|^s\chi} - T_{|z|^s\chi} T_f \in \mathcal{I}^{s/2m-1/2}.$$

Remark. No longer true for $H_{|z|^s\chi}$ replaced by H_f, $f = O(|z|^s)$.

Proof of Theorem A

Theorem A. *Let*

$$w(z) = e^{-|z|^{2m}}, \qquad m > 0.$$

Assume that $f, g \in L^\infty(\mathbf{C})$ *have the form*

$$f(z) = \sum_{j=0}^{q} |z|^{-j} f_j\Big(\frac{z}{|z|}\Big) + O\Big(\frac{1}{|z|^{q+1}}\Big) \qquad \text{as } |z| \to +\infty$$

and similarly for g, *where* $q + 1 > m$ *and* $f_j, g_j \in C^\infty(\mathbf{T})$.
Extend f_0 *to* $\mathbf{C} \setminus \{0\}$ *by* $f_0(z) := f_0\Big(\frac{z}{|z|}\Big)$ *and similarly for* g_0.
Then $H_{\overline{f}}, H_g \in \mathcal{I}^{-1/2}$, $H_{\overline{f}}^* H_g \in \mathcal{S}^{\mathrm{Dixm}}$ *belongs to the Dixmier class, is measurable, the limit*

$$Q(f,g)(e^{i\theta}) := \lim_{r \to +\infty} r^2 \partial f_0(re^{i\theta}) \overline{\partial} g_0(re^{i\theta})$$

exists for any $e^{i\theta} \in \mathbf{T}$, *and*

$$\mathrm{Tr}_\omega(H_{\overline{f}}^* H_g) = \frac{1}{2\pi m} \int_0^{2\pi} Q(f,g)(e^{i\theta})\, d\theta.$$

Step 1:

Extend f_j, g_j also by 0-homogeneity, and denote by f_{q+1}, g_{q+1} the remainder terms.

First of all, it is enough to prove the theorem for $f = f_0$, $g = g_0$. Indeed, if we know that

$$H_{\overline{f}_j}, H_{g_j} \in \mathcal{I}^{-1/2},$$

then from the observations above we easily get that

$$H^*_{\overline{f}} H_g - H^*_{\overline{f}_0} H_{g_0} \in \mathcal{S}^1.$$

Thus $H^*_{\overline{f}} H_g$ also belongs to $\mathcal{S}^{\mathrm{Dixm}}$ and has the same Tr_ω as $H^*_{\overline{f}_0} H_{g_0}$.

From now on, we thus assume that f, g are homogeneous of degree 0 on $\mathbf{C}$.

Step 2: Let

$$f(e^{i\theta}) = \sum_{j\in\mathbf{Z}} \hat{f}_j e_j, \qquad e_j(e^{i\theta}) := e^{ji\theta},$$

be the Fourier expansion of f. Since $f \in C^\infty(\mathbf{T})$ by hypothesis, we have

$$|\hat{f}_j| \leq \frac{C_f}{(j^2+1)^2}$$

by Cauchy estimates, so the series converges uniformly. Similarly for g. Hence

$$Q(f,g) = \sum_{j,l} \hat{f}_j \hat{g}_l Q(e_j, e_l).$$

If we show that

$$(*) \qquad \|H^*_{e_j} H_{e_l}\|_{\mathrm{Dixm}} \le C_m |jl|$$

for some constant C_m depending only on m, then also

$$\sum_{j,l} \|\hat{f}_j \hat{g}_l H^*_{\overline{e_j}} H_{e_l}\|_{\mathrm{Dixm}} \le C_f C_g C_m \sum_{j,l} \frac{|jl|}{(j^2+1)^2(l^2+1)^2} < \infty,$$

implying that $H^*_{\overline{f}} H_g \in \mathcal{S}^{\mathrm{Dixm}}$ and

$$\mathrm{Tr}_\omega(H^*_{\overline{f}} H_g) = \sum_{j,l} \hat{f}_j \hat{g}_l \,\mathrm{Tr}_\omega(H^*_{\overline{e_j}} H_{e_l}).$$

It is thus enough to prove Theorem A for $f = e_j$, $g = e_l$, $j, l \in \mathbf{Z}$, together with the norm estimate (*).

We start with the latter.

Step 3: From the general inequality

$$\|AB\|_{\mathrm{Dixm}} \leq \|AB\|_{-1} \leq 2\|A\|_{-1/2}\|B\|_{-1/2} = 2\sqrt{\|A^*A\|_{-1}\|B^*B\|_{-1}}$$

we see that it is enough to prove (*) for $j + l = 0$, i.e.

$$\|H_{e_l}^* H_{e_l}\|_{-1} \leq C_m l^2.$$

Now the last operator is diagonalized by the standard monomial basis: $H_{e_l}^* H_{e_l} z^k = d_k z^k$, where

$$d_k = \begin{cases} 1 & k + l < 0, \\ 1 - \dfrac{c_{k+l/2}^2}{c_k c_{k+l}} & k + l \geq 0. \end{cases}$$

This reduces (*) to showing that

$$x\left(1 - \frac{\Gamma(x + \frac{a}{2})^2}{\Gamma(x)\Gamma(x+a)}\right) \leq C_m a^2 \qquad \forall a \geq 0, \forall z \geq \frac{1}{m}.$$

Verified using properties of the Gamma function.

It remains to compute the Dixmier trace of $H_f^* H_g$ for $f = e_j$, $g = e_l$.

Step 4: Consider the unitary operator

$$Uf(z) := f(\epsilon z).$$

where $|\epsilon| = 1$. Then

$$U^* H^*_{\overline{f}} H_g U = H^*_{\overline{Uf}} H_{Ug}.$$

In particular,

$$U^* H^*_{\overline{e_j}} H_{e_l} U = \epsilon^{-j-l} H^*_{\overline{e_j}} H_{e_l}.$$

Since Dixmier trace is invariant under unitary maps and $\epsilon \in \mathbf{T}$ can be taken arbitrary, it follows that

$$\mathrm{Tr}_\omega(H^*_{\overline{e_j}} H_{e_l}) = 0 \qquad \text{if } j + l \neq 0.$$

When $j+l=0$, we saw that $H^*_{e_j} H_{e_l}$ is diagonal with explicitly given eigenvalues d_k, giving

$$\operatorname{Tr}_\omega(H^*_{e_l} H_{e_l}) = \lim_{k\to\infty} k d_k = \cdots = \frac{l^2}{4m}.$$

On the other hand, direct computation gives

$$\frac{1}{2\pi m} \int_0^{2\pi} Q(e_j, e_l)(e^{i\theta})\, d\theta = -\delta_{j+l,0} \frac{jl}{4m}.$$

Thus the left-hand side is equal to $\operatorname{Tr}_\omega(H^*_{e_l} H_{e_l})$ for any $j, l \in \mathbf{Z}$, proving the last claim and hence Theorem A. $\square$

Proof of Theorem B

Theorem B. *Let*

$$w(z) = \rho(z)|z|^q e^{-|z|^2}$$

where $q \in \mathbf{R}$ *and*

$$\rho(z) \qquad \text{is bounded and bounded away from 0 as } z \to \infty.$$

Then

(i) $f \in \mathcal{A}^s,\ s < 0 \implies T_f \in \mathcal{S}^{-2/s,\infty}$.

(ii) $f \in \mathcal{A}^s,\ s < -2/p \implies T_f \in \mathcal{S}^p\ (p \geq 1)$.

(iii) $f, g \in \mathcal{A}^0 \implies H_f^* H_g \in \mathcal{S}^{\mathrm{Dixm}}$ *and* Tr_ω *is given by the same formula as for* $w(z) = e^{-|z|^2}$.

Quite generally, consider Toeplitz operators T_f on some weighted Fock space $\mathcal{F}_w$, and the Toeplitz operators $T_f^{(\rho)}$ on the weighted Fock space $\mathcal{F}_{\rho w}$, where ρ is a positive function.

We assume that both spaces contain a common dense subset (e.g. polynomials).

Proposition. (i) *The operator T_ρ is (possibly unbounded) densely defined, selfadjoint and positive (i.e. $\langle T_\rho f, f\rangle \geq 0 \; \forall f \in \operatorname{dom} T_\rho$), hence has an inverse T_ρ^{-1} with the same properties.*

(ii) *The positive square root $T_\rho^{1/2}$ of T_ρ extends by continuity to a unitary isomorphism of $\mathcal{F}_{\rho w}$ onto $\mathcal{F}_w$.*

(iii) *For any $f \in L^\infty(\mathbf{C})$, we have under this isomorphism*

$$T_f^{(\rho)} \cong T_\rho^{-1/2} T_{\rho f} T_\rho^{-1/2}.$$

Proof of Theorem B: Set $w(z) = e^{-|z|^2}$ in the last Proposition.

By (iii), $T_f^{(\rho)}$ belongs to some unitary ideal like $\mathcal{S}^p$, $\mathcal{I}^{-s}$, etc., if and only if $T_\rho^{-1/2} T_{\rho f} T_\rho^{-1/2}$ does.

Using the relationship between the Toeplitz operators on $\mathcal{F}_1$ and the Weyl operators on $L^2(\mathbf{R})$, this reduces the problem again to the one of membership of (sums of products of) Weyl operators in these ideals, which are handled by standard machinery for ΨDOs. $\square$

More details in:

- H. Bommier-Hato, M. Engliš, E.-H. Youssfi: *Dixmier trace and the Fock space,* `http://www.math.cas.cz/englis/79.pdf`

  (the case of standard Fock space; to appear in Bull. Sci. Math.)

- H. Bommier-Hato, M. Engliš, E.-H. Youssfi: *Dixmier classes on generalized Segal-Bargmann-Fock spaces,* `http://www.math.cas.cz/englis/SBargm.pdf`

  (Theorems A+B, Fact C)

THANKS FOR YOUR ATTENTION!