Toeplitz and Hankel operators on weighted Fock spaces

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Abstract. We give criteria for the membership of Toeplitz operators and of products of Hankel operators, with symbols of a certain type, in Schatten ideals and in the Dixmier class, and formulas for their Dixmier trace, on a variety of weighted Segal-Bargmann-Fock spaces on the complex plane.

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**HARDY SPACE OPERATORS**

$$T = \{ z \in \mathbb{C} : |z| = 1 \}$$

**Hardy space:** \( L^2(T) \supset H^2 = \{ f : \hat{f}(n) = 0 \ \forall n < 0 \} = L^2_{\text{hol}}(T), \)

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^1 f(e^{i\theta}) e^{-ni\theta} \, d\theta.$$  

**Szegö projection:** \( S : L^2(T) \rightarrow H^2. \)

**Toeplitz operator** with symbol \( f \in L^\infty(T): \)

$$T_f^{\text{Hardy}} : H^2 \rightarrow H^2, \quad g \mapsto S(fg)$$

**Hankel operator:**

$$H_f^{\text{Hardy}} : H^2 \rightarrow L^2 \ominus H^2 = \overline{H^2_0}, \quad g \mapsto (I - S)(fg).$$
**Bergman space operators**

\[ D = \{ z \in \mathbb{C} : |z| < 1 \} \]

**Bergman space**: \( L^2(D) \supset A^2 = \{ f : f \text{ holomorphic on } D \} \)

**Bergman projection**: \( P : L^2(D) \to A^2 \)

**Toeplitz operator** with symbol \( f \in L^\infty(D) \):

\[ T_f : A^2 \to A^2, \quad g \mapsto P(f g); \]

**Hankel operator**:

\[ H_f : A^2 \to L^2(D) \ominus A^2, \quad g \mapsto (I - P)(f g). \]
Similarly — on the ball $B^n \subset C^n$, or any $\Omega \subset C^n$:

Bergman space (wrt the Lebesgue measure)

Hardy space (Poisson extension holomorphic; wrt surface measure)
   [bigger orthocomplement]

Szegö/Bergman projection, Toeplitz ops., Hankel ops.
**$H_f$ in Schatten Classes**

Hankel operators on $H^2(\mathbf{T})$ [Peller 1982] [Semmes 1984]:
for $f$ holomorphic, $H_{\bar{f}} \in S^p \iff f \in B^p$, the $p$-th Besov space:

$$\int_{\mathbf{D}} |f^{(k)}(z)|^p (1 - |z|^2)^{kp-2} \, dm(z) < \infty,$$

for some ($\iff$ any) $k > 1/p$. \hspace{1cm} (0 < p < \infty) \hspace{1cm} (Trivial for $f \in C^\infty(\mathbf{T})$.)

Hankel operators on $A^2(\mathbf{D})$ [Arazy-Fisher-Janson-Peetre 1988]:
for $f$ holomorphic, $H_{\bar{f}} \in S^p$, $1 < p < \infty$, $\iff f \in B^p$;

$H_{\bar{f}} \in S^p$, $p \leq 1$, $\iff H_{\bar{f}} = 0$.

\ldots **Cut-off** at $p = 1$. \hspace{1cm} (Never $S^1$ even for $f \in C^\infty(\overline{\mathbf{D}})$.)
Hankel operators on $A^2(\mathbb{B}^n)$, $n > 1$ [Arazy-Fisher-Peetre 1988]:
for $f$ holomorphic, $H_{\overline{f}} \in \mathcal{S}^p$, $2n < p < \infty$, $\iff f \in \mathcal{B}^p$;
$H_{\overline{f}} \in \mathcal{S}^p$, $p \leq 2n$, $\iff H_{\overline{f}} = 0$.
(cut-off at $p = 2n$)

Same result — for any $\Omega$ smoothly bounded strictly pseudoconvex in $\mathbb{C}^n$, $n \geq 2$. [Li-Luecking 1995]

At the cutoff: Dixmier ideal and Dixmier trace.
The Dixmier ideal

Recall: a compact operator $T$ on a Hilbert space belongs to $S^p$ iff

$$\sum_j s_j(T)^p < \infty,$$

where $s_0(T) \geq s_1(T) \geq s_2(T) \geq \ldots$ are the eigenvalues of $(T^*T)^{1/2}$ (counting multiplicities).

Schatten ideals: $T \in S^p \iff \{s_j(T)\} \in \ell^p \quad (0 < p < \infty)$;

$$T \in S^{p,\infty} \iff s_j(T) = O(j^{-1/p}) \quad (0 < p < \infty).$$

($S^1$ — trace class).

Dixmier ideal $S^{\text{Dixm}}$: consists of all $T$ such that

$$\sum_{j=1}^N s_j(T) = O(\log N).$$

Norm:

$$\|T\|_{\text{Dixm}} := \sup_{N \geq 2} \frac{1}{\log N} \sum_{j=0}^N s_j(T).$$

Inclusions: $S^1 \subset S^{1,\infty} \subset S^{\text{Dixm}} \subset S^p, \quad \forall p > 1.$
**Dixmier trace**

\[ \omega : l^\infty \to \mathbb{C} \text{ a Banach limit} \]

i.e. \( \omega \in (l^\infty)^* \) of norm 1 extending \( \lim \in c^* \) on \( c \subset l^\infty \)

\( \omega \) is **scaling invariant** if \( \omega(x_1, x_1, x_2, x_2, \ldots) = \omega(x_1, x_2, \ldots) \)

**Dixmier trace** of \( A \in S^{\text{Dixm}}, A \geq 0 \):

\[
\operatorname{Tr}_\omega(A) := \omega \left( \frac{1}{1 + \log n} \sum_{j=1}^{n} s_j(A) \right).
\]

One has \( \operatorname{Tr}_\omega(A + B) = \operatorname{Tr}_\omega(A) + \operatorname{Tr}_\omega(B) \) if \( \omega \) scaling invariant; hence, makes sense to set

\[
\operatorname{Tr}_\omega(A) = \operatorname{Tr}_\omega(A_+) - \operatorname{Tr}_\omega(A_-) \quad \text{for} \quad A = A_+ - A_- = A^* \in S^{\text{Dixm}},
\]

\[
\operatorname{Tr}_\omega(A) = \operatorname{Tr}_\omega \left( \frac{A + A^*}{2} \right) + i \operatorname{Tr}_\omega \left( \frac{A - A^*}{2i} \right) \quad \text{for arbitrary} \quad A \in S^{\text{Dixm}}.
\]

\( A \) is **measurable** if \( \operatorname{Tr}_\omega(A) \) is the same for all \( \omega \).
Properties of $\text{Tr}_\omega$:

- Cyclicity: $\text{Tr}_\omega(AB) = \text{Tr}_\omega(BA)$ \quad $A \in \mathcal{S}^{\text{Dixm}}$, $B$ bounded.

- Nontrivial: $\text{Tr}_\omega(A) = 0$ if $A$ trace-class.

- for $T \geq 0$

$$
\text{Tr}_\omega(T) = \lim_{N \to \infty} \frac{1}{\log N} \sum_{j=0}^{N} s_j(T)
$$

if the limit exists.

[Connes – NG book 1994]
We had the cut-off phenomena: for $f$ holomorphic on the disc,

$$H \tilde{f} \in S^p \iff \begin{cases} f \in B^p, & 1 < p < \infty, \\ f = \text{const.}, & 0 < p \leq 1, \end{cases}$$

and $f$ holomorphic on the ball or on $s$-pscpx $\Omega \subset \mathbb{C}^n$ $s$-pscpx, $n \geq 2$,

$$H \tilde{f} \in S^p \iff \begin{cases} f \in B^p, & 2n < p < \infty, \\ f = \text{const.}, & 0 < p \leq 2n. \end{cases}$$

**Question:** When is

$$H_f \in S^{\text{Dixm}} \quad \text{on } \mathbb{D},$$

$$H_{f_1}^* H_{f_2} \ldots H_{f_{2n-1}}^* H_{f_{2n}} \in S^{\text{Dixm}} \quad \text{on } \mathbb{B}^n, n \geq 2,$$

or on $\Omega \subset \mathbb{C}^n$, $n \geq 2$, nice?

What are the corresponding Dixmier traces

$$\text{Tr}_\omega(|H_f|), \quad \text{Tr}_\omega(H_{f_1}^* H_{f_2} \ldots H_{f_{2n-1}}^* H_{f_{2n}})?$$
Known previously:

- for $\mathbb{D}$ [Arazy-Fisher-Peetre 1988]

\[ f \in B^1 \implies H_f \in S^{\text{Dixm}}, \]

but not conversely.

- $0 \neq H_f \in \mathcal{I}$, a unitary ideal $\implies S^{\text{Dixm}} \subset \mathcal{I}$.

- Nothing known about $\text{Tr}_\omega(|H_f|)$.

- Nothing known for $\mathcal{B}^n$, $n \geq 2$, or strictly-pseudoconvex.
**Results — disc**

**Theorem.** [R. Rochberg, ME] If $f \in C^\infty(\overline{D})$, then

$$\text{Tr}_\omega(|H_f|) = \int_\mathcal{T} |\bar{\partial}f| \, d\sigma,$$

where $d\sigma = \text{normalized arc-length}$.

In particular: $H_f$ measurable. (Nontrivial even for smooth $f$.)

**Theorem.** [RR, ME] For $f$ holomorphic on $D$, TFAE:

1. $f' \in H^1$;
2. $H_{\overline{f}} \in S^{1,\infty}$;
3. $H_{\overline{f}} \in S^{\text{Dixm}}$.

In that case $|H_{\overline{f}}|$ is measurable and

$$\text{Tr}_\omega(|H_{\overline{f}}|) = \int_\mathcal{T} |f'| \, d\sigma = \|f'\|_{H^1}.$$
Results — several complex variables

**Theorem.** $\Omega \subset \mathbb{C}^n$ smoothly bounded strictly pseudoconvex. Then for any $2n$ functions $f_1, g_1, \ldots, f_n, g_n \in C^\infty(\overline{\Omega})$,

$$H_{f_1}^* H_{g_1} \cdots H_{f_n}^* H_{g_n} =: H \in S^{\text{Dixm}}$$

and

$$\text{Tr}_\omega(H) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \prod_{j=1}^n \mathcal{L}((\partial_b g_j, \partial_b f_j)) \, d\mu,$$

where

$$d\mu := \frac{1}{2i^n} (\partial r - \bar{\partial} r) \wedge (\partial \partial r)^{n-1}.$$

Here $\mathcal{L}$ is the dual Levi form on $T''^*$ (the dual of the anti-holomorphic complex tangent space on $\partial\Omega$), and $r$ is a defining function for $\Omega$.

In particular, $H$ is measurable.
Results — the Fock Space

Fock (Segal-Bargmann) space:

\[ \mathcal{F}_\gamma := \{ f \in L^2(\mathbb{C}^n, e^{-\gamma|z|^2} (\frac{\gamma}{\pi})^n \, dz) : f \text{ entire} \}, \quad \gamma > 0. \quad (\gamma = \frac{1}{2}) \]

Toeplitz and Hankel operators:

\[ T_f : u \mapsto P(fu), \quad H_f : u \mapsto (I - P)(fu), \]

where \( f \in L^\infty(\mathbb{C}^n) \) and \( P : L^2 \to \mathcal{F}_\gamma \) is the OG projection.

Analogue of \( C^\infty \) on the closure:

\[ f \in \mathcal{A} \quad \overset{\text{def}}{\iff} \quad f(z) \approx \sum_{j=0}^{\infty} f_j(z) \quad \text{as } |z| \to +\infty, \]

where \( f_j(z) \) is homogeneous of degree \(-j\), i.e. \( f_j(tz) = t^{-j} f_j(z) \forall t > 0 \).

(Symbol classes for psdo's.)
\textbf{Theorem.} For $f, g \in \mathcal{A}$ and $\zeta \in S^{2n-1}$, denote

$$Q(f, g) := \lim_{r \to +\infty} r^2 \sum_{j=1}^{n} \partial_j f(r\zeta) \cdot \overline{\partial_j g(r\zeta)}$$

(the limits exists thanks to the definition of $\mathcal{A}$, and in fact we may replace $f, g$ by their top-degree components $f_0, g_0$).

Then for any $f_1, g_1, \ldots, f_n, g_n \in \mathcal{A}$, the product

$$H^*_{f_1} H_{g_1} \ldots H^*_{f_n} H_{g_n} =: H$$

belongs to $S^{\text{Dixm}}$, is measurable, and

$$\text{Tr}_\omega(H) = \frac{1}{n!} \int_{S^{2n-1}} Q(f_1, g_1) \cdots Q(f_n, g_n) \, d\sigma$$

where $d\sigma$ is the normalized surface measure on $S^{2n-1}$. 
Proof. Weyl operator with symbol \( a = a(x, \xi) \) on \( L^2(\mathbb{R}^n) \):

\[
W_a f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a \left( \frac{x+y}{2}, \xi \right) e^{-i(x-y,\xi)} f(y) \, dy \, d\xi.
\]

Converges if \( a \in \mathcal{S}(\mathbb{R}^{2n}) \); extends in standard way to \( a \in \mathcal{S}'(\mathbb{R}^{2n}) \).

Special cases: \( W_a(x) = M_a \); \( W_{\xi^\alpha} = i^\alpha \partial^\alpha \).

Bargmann transform: isometry \( \beta : L^2(\mathbb{R}^n) \to \mathcal{F}_{-1/2} \),

\[
\beta f(z) := \frac{1}{(4\pi^3)^{n/4}} \int_{\mathbb{R}^n} f(x) e^{x\cdot z - x\cdot x/2 - z\cdot z/4} \, dx.
\]

(Extends also to general \( F_\gamma \) via dilations \( z \mapsto \sqrt{\gamma} z \).)

Fact: Relationship to Toeplitz operators:

\[
\beta^* T_a \beta = W_{\mathcal{E}a}
\]

where \( a(x, \xi) \) is also viewed as the function \( a(z) \) of \( z = x + i\xi \in \mathbb{C}^n \), and

\[
\mathcal{E}a(z) = \left(\frac{2\gamma}{\pi}\right)^n \int_{\mathbb{C}^n} a(w) e^{-2\gamma|z-w|^2} \, dw = e^{\Delta/8\gamma} a(z)
\]

is the heat solution at time \( t = \frac{1}{8\gamma} \).

Reduces the problem to deciding when \( W_{\mathcal{E}(f \circ g)} - W_{\mathcal{E} f} W_{\mathcal{E} g} \in \mathcal{S}^{2n, \infty} \).
Using the interplay between $W_a$ and $T_a = W_{\varepsilon a} = W_{a+\text{LOT}}$:

**Theorem.**

(a) Let $p > 1$ and $a \in A^m$, $m < -2n/p$. Then $W_a, T_a \in S^p$.
(b) Let $p > 1$ and $a \in A^m$, $m \leq -2n/p$. Then $W_a, T_a \in S^{p,\infty}$.
(c) If $a \in A^0$, then $W_a$ is bounded.

(The last usually proved using Calderon-Vaillancourt — we are able to prove it using only that $T_f$ is bounded for $f$ bounded.)

Known for $W_a$; folk lore for Toeplitz.

Here $A^m$ denotes the class of $f$ with homogeneous expansion

$$f(\bar{z}) \approx |z|^m \sum_{j=0}^{\infty} f_j(z) \quad \text{as } |z| \to +\infty.$$
Rest of this talk: Generalizations to weighted Fock spaces.
Notation:

\[ \mathcal{F}_w := \{ f \in L^2(\mathbb{C}^n, w) : f \text{ is holomorphic} \} . \]

Here \( w \) is (positive continuous) assumed to be such that

\[ |z|^k w(z) \text{ is integrable for all } k \geq 0, \]

so that polynomials belong to \( \mathcal{F}_w \). In all cases we will consider here, they will also be dense in \( \mathcal{F}_w \).

Example:

\[ \mathcal{F}_m := \mathcal{F}_w \quad \text{for} \quad w(z) = e^{-|z|^{2m}}, \]

“higher-order Fock” spaces. \( \square \)

Toeplitz and Hankel operators on \( \mathcal{F}_w \):

\[ T_f = P_+ M_f, \quad H_f = P_- M_f, \]

where \( P_+ : L^2(\mathbb{C}^n, w) \to \mathcal{F}_w \) is the orthogonal projection, \( P_- = I - P_+ \), and \( M_f : u \mapsto fu \) is the operator of “multiplication by \( f \)”.

Question: Membership in \( \mathcal{S}^p, \mathcal{S}^{\text{Dixm}}, \text{Tr}_\omega \).

Will work on \( \mathbb{C} \); the case of \( \mathbb{C}^n \) involves just more technicalities.
**Related work:**

- [Holland, Rochberg 2001] — radial weights, estimates for Bergman kernel, Hankel forms
- [Bommier-Hato, Youssfi 2007] — $H_f$, $f$ holomorphic, on $\mathcal{F}_m$
- [Seip, Youssfi 2012] — $S^p$ criteria for $H_f$, $f$ holomorphic, similar (finer) estimates for Bergman kernel
- [Lin, Rochberg 1995], [Constantin, Ortega-Cerda 2011] (for $\overline{\partial}$), …

Nothing known for $S^{Dixm}$, $Tr_\omega$. 
Results

Theorem A. Let

\[ w(z) = e^{-|z|^{2m}}, \quad m > 0. \]

Assume that \( f, g \in L^\infty(\mathbb{C}) \) have the form

\[ f(z) = \sum_{j=0}^{q} |z|^{-j} f_j \left( \frac{z}{|z|} \right) + O \left( \frac{1}{|z|^{q+1}} \right) \quad \text{as} \ |z| \to +\infty \]

and similarly for \( g \), where \( q + 1 > m \) and \( f_j, g_j, j = 0, 1, \ldots, q \), are some functions in \( C^\infty(\mathbb{T}) \).

Extend \( f_0 \) to \( \mathbb{C} \setminus \{0\} \) by \( f_0(z) := f_0 \left( \frac{z}{|z|} \right) \) and similarly for \( g_0 \).
Then $H_f$, $H_g$ belong to $S^{2,\infty}$, the operator $H_f^* H_g$ belongs to the Dixmier class, is measurable, the limit

$$Q(f, g)(e^{i\theta}) := \lim_{r \to +\infty} r^2 \partial f_0(re^{i\theta}) \overline{\partial g_0(re^{i\theta})}$$

exists for any $e^{i\theta} \in \mathbf{T}$, and

$$\text{Tr}_\omega(H_f^* H_g) = \frac{1}{2\pi m} \int_0^{2\pi} Q(f, g)(e^{i\theta}) \, d\theta.$$ 

Also an analogous result for $T_f$.

The hypothesis $q + 1 > m$ is essential: for $f(z) = e^{i|z|^2}$, $H_f$ is not even compact.
Theorem B. Let

\[ w(z) = \rho(z)|z|^q e^{-|z|^2} \]

where \( q \in \mathbb{R} \) and

\[ \rho(z) \quad \text{is bounded and bounded away from 0 as } z \to \infty. \]

Then

(i) \( f \in \mathcal{A}^s, \ s < 0 \implies T_f \in \mathcal{S}^{-2/s, \infty} \).

(ii) \( f \in \mathcal{A}^s, \ s < -2/p \implies T_f \in \mathcal{S}^p \ (p \geq 1) \).

(iii) \( f, g \in \mathcal{A}^0 \implies H_f^* H_g \in \mathcal{S}^{\text{Dixm}} \) and \( \text{Tr}_\omega \) is given by the same formula as for \( w(z) = e^{-|z|^2} \).

Proof of Theorem A — elementary methods.

Proof of Theorem B — reduction to Weyl calculus.
Fact C. Let
\[ w(z) = e^{-|z|^{2m}}, \quad m > 0. \]

Then there exists a natural “Bargman transform”
\[ \beta_m : L^2(\mathbb{R}) \to F_m \]
such that
\[ \beta^*_m T_f \beta_m = W \mathcal{E}_m f \]
where \( \mathcal{E}_m \) is a certain \( \Psi \)DO on \( \mathbb{C} \).

We originally hoped to use this for giving a proof of Theorem A also by reduction to the Weyl calculus.
However, \( \mathcal{E}_m \) is too complicated.
Proof of Theorem A

For convenience, introduce the notation \((s > 0)\)

\[
\mathcal{I}^{-s} = S^{1/s, \infty} = \{ T : s_j(T) = O(j^{-s}) \text{ as } j \to \infty \}.
\]

Then \(\mathcal{I}^{-s}\) is a vector space, with quasi-norm

\[
\|T\|_{-s} := \sup_j (j + 1)^s s_j(T)
\]

satisfying

\[
\|A + B\|_{-s} \leq 2^s (\|A\|_{-s} + \|B\|_{-s}),
\]

\[
\|AB\|_{-s-t} \leq 2^{s+t} \|A\|_{-s} \|B\|_{-t}.
\]

In particular, \(\mathcal{I}^{-s} \cdot \mathcal{I}^{-t} \subset \mathcal{I}^{-s-t}\).

Let also \(\chi\) denote the characteristic function of the exterior disc \(|z| \geq 1\).

Then the functions

\[
|z|^s \chi(z)
\]

are defined for any \(s \in \mathbb{R}\) (we avoid the singularity \(z = 0\) for negative \(s\)).
Observation 1: \( T|z|^s\chi \in \mathcal{I}^{s/2m} \).

**Proof.** For any radial function \( w \) on \( \mathbb{C} \), denote by
\[
c_k(w) := \int_{\mathbb{C}} |z|^{2k} w(z) \, dz
\]
its moments. Then if \( \phi \) is another radial function, the Toeplitz operator \( T_\phi \) on \( \mathcal{F}_w \) is diagonalized by the monomial basis \( \{z^k\} \):
\[
T_\phi : z^k \mapsto \frac{c_k(\phi w)}{c_k(w)} z^k.
\]
Applying this to \( w(z) = e^{-|z|^{2m}} \), \( \phi(z) = |z|^s\chi(z) \) gives the result, since
\[
c_k(w) = \frac{\pi}{m} \Gamma\left(\frac{m + 1}{\pi}\right), \quad c_k(\phi w) \sim c_{k+s/2}(w),
\]
and
\[
\frac{c_{k+s/2}(w)}{c_k(w)} \sim \frac{\Gamma\left(\frac{k+s+1}{2m}\right)}{\Gamma\left(\frac{k+1}{m}\right)} \sim k^{s/2m}
\]
by Stirling’s formula. \( \square \)
Observation 2: $T_f \in \mathcal{I}^{s/2m}$ if $f(z) = O(|z|^s)$ as $z \to \infty$.

Proof. Write

$$T_f = T(1-\chi)f + T|z|^s\chi g$$

with $g$ bounded. Then

$$T|z|^s\chi g = P + M|z|^s\chi g P_+ = (M|z|^{s/2}\chi P_+)^* Mg(M|z|^{s/2}\chi P_+)$$

while

$$(M|z|^{s/2}\chi P_+)^*(M|z|^{s/2}\chi P_+) = T|z|^s\chi.$$ 

By previous observation, the last product belongs to $\mathcal{I}^{s/2m}$, hence $(M|z|^{s/2}\chi P_+) \in \mathcal{I}^{s/4m}$ and $T|z|^s\chi g \in \mathcal{I}^{s/2m}$.

A similar argument shows that $T(1-\chi)f$ in fact belongs to all $\mathcal{I}^s$, $s < 0$. So $T_f \in \mathcal{I}^{s/2m}$. □
Using the formula
\[ T_f g - T_f T_g = H^*_f H_g, \]
similar argument yields also the following two observations.

**Observation 3:** \( H_f \in \mathcal{I}^{s/2m} \) if \( f(z) = O(|z|^s) \) as \( z \to \infty \).

**Observation 4:** For \( f \in L^\infty(\mathbb{C}) \) and \( s \leq 0 \),
\[ H_{|z|^s \chi}, \quad T_{f|z|^s \chi} - T_f T_{|z|^s \chi}, \quad T_{f|z|^s \chi} - T_{|z|^s \chi} T_f \in \mathcal{I}^{s/2m - 1/2}. \]

**Remark.** No longer true for \( H_{|z|^s \chi} \) replaced by \( H_f, \ f = O(|z|^s) \).
Proof of Theorem A

Theorem A. Let

\[ w(z) = e^{-|z|^{2m}}, \quad m > 0. \]

Assume that \( f, g \in L^\infty(\mathbb{C}) \) have the form

\[ f(z) = \sum_{j=0}^{q} |z|^{-j} f_j \left( \frac{z}{|z|} \right) + O \left( \frac{1}{|z|^{q+1}} \right) \quad \text{as} \quad |z| \to +\infty \]

and similarly for \( g \), where \( q + 1 > m \) and \( f_j, g_j \in C^\infty(\mathbb{T}) \).

Extend \( f_0 \) to \( \mathbb{C} \setminus \{0\} \) by \( f_0(z) := f_0 \left( \frac{z}{|z|} \right) \) and similarly for \( g_0 \).

Then \( H_f, H_g \in \mathcal{I}^{-1/2}, \ H_f^* H_g \in S^{\text{Dixm}} \) belongs to the Dixmier class, is measurable, the limit

\[ Q(f, g)(e^{i\theta}) := \lim_{r \to +\infty} r^2 \partial f_0(r e^{i\theta}) \overline{\partial g_0(r e^{i\theta})} \]

exists for any \( e^{i\theta} \in \mathbb{T} \), and

\[ \text{Tr}_w(H_f^* H_g) = \frac{1}{2\pi m} \int_0^{2\pi} Q(f, g)(e^{i\theta}) \, d\theta. \]
Step 1:

Extend $f_j, g_j$ also by 0-homogeneity, and denote by $f_{q+1}, g_{q+1}$ the remainder terms.

First of all, it is enough to prove the theorem for $f = f_0, g = g_0$. Indeed, if we know that

$$H_{\bar{f}} f_j, H_{\bar{g}} g_j \in \mathcal{I}^{-1/2},$$

then from the observations above we easily get that

$$H_{\bar{f}}^{*} H_f - H_{\bar{f}_0}^{*} H_{f_0} \in S^1.$$

Thus $H_{\bar{f}}^{*} H_g$ also belongs to $S^{\text{Dixm}}$ and has the same Tr$\omega$ as $H_{\bar{f}_0}^{*} H_{g_0}$.

From now on, we thus assume that $f, g$ are homogeneous of degree 0 on $\mathbb{C}$. 
Step 2: Let
\[ f(e^{i\theta}) = \sum_{j \in \mathbb{Z}} \hat{f}_j e_j, \quad e_j(e^{i\theta}) := e^{ji\theta}, \]
be the Fourier expansion of \( f \). Since \( f \in C^\infty(T) \) by hypothesis, we have
\[ |\hat{f}_j| \leq \frac{C_f}{(j^2 + 1)^2} \]
by Cauchy estimates, so the series converges uniformly. Similarly for \( g \). Hence
\[ Q(f, g) = \sum_{j, l} \hat{f}_j \hat{g}_l Q(e_j, e_l). \]
If we show that

\[(*) \quad \|H_{e_j}^*H_{e_l}\|_{Dixm} \leq C_m|j||l|\]

for some constant $C_m$ depending only on $m$, then also

\[
\sum_{j,l} \|\hat{f}_j\hat{g}_lH_{e_j}^*H_{e_l}\|_{Dixm} \leq C_fC_gC_m \sum_{j,l} \frac{|j||l|}{(j^2 + 1)^2(l^2 + 1)^2} < \infty,
\]

implying that $H_{\hat{f}}^*H_g \in S^{Dixm}$ and

\[
\text{Tr}_\omega(H_{\hat{f}}^*H_g) = \sum_{j,l} \hat{f}_j\hat{g}_l \text{Tr}_\omega(H_{e_j}^*H_{e_l}).
\]

It is thus enough to prove Theorem A for $f = e_j$, $g = e_l$, $j, l \in \mathbb{Z}$, together with the norm estimate $(*)$.

We start with the latter.
Step 3: From the general inequality
\[ \|AB\|_{\text{Dixm}} \leq \|AB\|_1 - 1 \leq 2\|A\|_{-1/2}\|B\|_{-1/2} = 2\sqrt{\|A^*A\|_{-1}\|B^*B\|_{-1}} \]
we see that it is enough to prove (*) for \( j + l = 0 \), i.e.
\[ \|H_{j}^* H_{l} \|_{-1} \leq C m l^2. \]

Now the last operator is diagonalized by the standard monomial basis: \( H_{e_j}^* H_{e_l} z^k = d_k z^k \), where
\[
d_k = \begin{cases} 
1 & k + l < 0, \\
1 - \frac{c_k^2}{c_k c_{k+l}} & k + l \geq 0.
\end{cases}
\]

This reduces (*) to showing that
\[ x \left( 1 - \frac{\Gamma(x + \frac{a}{2})^2}{\Gamma(x)\Gamma(x + a)} \right) \leq C m a^2 \quad \forall a \geq 0, \forall z \geq \frac{1}{m}. \]

Verified using properties of the Gamma function.

It remains to compute the Dixmier trace of \( H_f^* H_g \) for \( f = e_j, g = e_l \).
Step 4: Consider the unitary operator

$$Uf(z) := f(\epsilon z).$$

where $|\epsilon| = 1$. Then

$$U^* H_f^* H_g U = H_f^{*} H_{Ug}. $$

In particular,

$$U^* H_{e_j}^* H_{e_l} U = \epsilon^{-j-l} H_{e_j}^{*} H_{e_l}. $$

Since Dixmier trace is invariant under unitary maps and $\epsilon \in \mathbf{T}$ can be taken arbitrary, it follows that

$$\text{Tr}_\omega(H_{e_j}^{*} H_{e_l}) = 0 \quad \text{if } j + l \neq 0.$$
When \( j + l = 0 \), we saw that \( H_{e_j}^* H_{e_l} \) is diagonal with explicitly given eigenvalues \( d_k \), giving

\[
\text{Tr}_\omega(H_{e_l}^* H_{e_l}) = \lim_{k \to \infty} kd_k = \cdots = \frac{l^2}{4m}.
\]

On the other hand, direct computation gives

\[
\frac{1}{2\pi m} \int_0^{2\pi} Q(e_j, e_l)(e^{i\theta}) \, d\theta = -\delta_{j+l,0} \frac{jl}{4m}.
\]

Thus the left-hand side is equal to \( \text{Tr}_\omega(H_{e_l}^* H_{e_l}) \) for any \( j, l \in \mathbb{Z} \), proving the last claim and hence Theorem A. \( \square \)
**Proof of Theorem B**

**Theorem B.** Let

\[ w(z) = \rho(z)|z|^qe^{-|z|^2} \]

where \( q \in \mathbb{R} \) and

\[ \rho(z) \text{ is bounded and bounded away from 0 as } z \to \infty. \]

Then

(i) \( f \in \mathcal{A}^s, s < 0 \implies T_f \in \mathcal{S}^{-2/s, \infty}. \)

(ii) \( f \in \mathcal{A}^s, s < -2/p \implies T_f \in \mathcal{S}^p \) (\( p \geq 1 \)).

(iii) \( f, g \in \mathcal{A}^0 \implies H_f^*H_g \in \mathcal{S}^{\text{Dixm}} \) and Tr\( \omega \) is given by the same formula as for \( w(z) = e^{-|z|^2} \).
Quite generally, consider Toeplitz operators $T_f$ on some weighted Fock space $\mathcal{F}_w$, and the Toeplitz operators $T_f^{(\rho)}$ on the weighted Fock space $\mathcal{F}_{\rho w}$, where $\rho$ is a positive function.

We assume that both spaces contain a common dense subset (e.g. polynomials).

**Proposition.** (i) The operator $T_\rho$ is (possibly unbounded) densely defined, selfadjoint and positive (i.e. $\langle T_\rho f, f \rangle \geq 0 \ \forall f \in \text{dom } T_\rho$), hence has an inverse $T_\rho^{-1}$ with the same properties.

(ii) The positive square root $T_\rho^{1/2}$ of $T_\rho$ extends by continuity to a unitary isomorphism of $\mathcal{F}_{\rho w}$ onto $\mathcal{F}_w$.

(iii) For any $f \in L^\infty(\mathbb{C})$, we have under this isomorphism

$$T_f^{(\rho)} \cong T_\rho^{-1/2} T_\rho f T_\rho^{-1/2}.$$
Proof of Theorem B: Set \( w(z) = e^{-|z|^2} \) in the last Proposition.

By (iii), \( T_f^{(\rho)} \) belongs to some unitary ideal like \( S^p, \mathcal{I}^{-s}, \) etc., if and only if \( T_\rho^{-1/2}T_\rho f T_\rho^{-1/2} \) does.

Using the relationship between the Toeplitz operators on \( \mathcal{F}_1 \) and the Weyl operators on \( L^2(\mathbb{R}) \), this reduces the problem again to the one of membership of (sums of products of) Weyl operators in these ideals, which are handled by standard machinery for \( \Psi \)DOs. \( \square \)
More details in:

  (the case of standard Fock space; to appear in Bull. Sci. Math.)

  (Theorems A+B, Fact C)

Thanks for your attention!