Cyclic Properties and Isometric Asymptotes of Tree-shift operators

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23th May 2013, Gargnano, HFS 2013

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Introduction

Weighted unilateral shifts: $\{e_n\}_{n=0}^{\infty}$ is an ONB in $\mathcal{H}$, $\{w_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ bounded weight-sequence

$$W \colon \mathcal{H} \to \mathcal{H}, \quad We_n = w_{n+1} e_{n+1}.$$

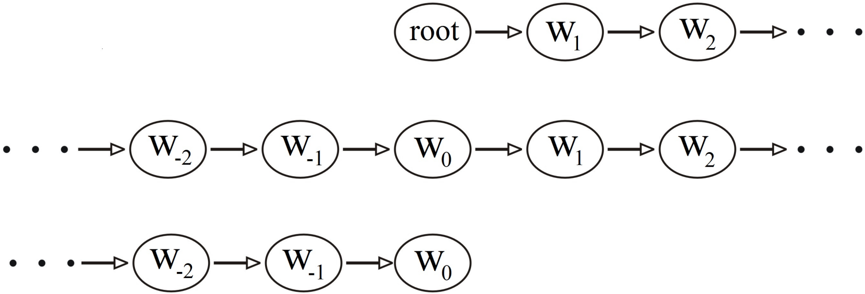
Weighted bilateral shifts: $\{e_n\}_{n=-\infty}^{\infty}$ is an ONB in $\mathcal{H}$, $\{w_n\}_{n=-\infty}^{\infty} \subseteq \mathbb{C}$ bounded weight-sequence

$$W \colon \mathcal{H} \to \mathcal{H}, \quad We_n = w_{n+1} e_{n+1}.$$

Weighted backward shifts: $\{e_n\}_{n=0}^{\infty}$ is an ONB in $\mathcal{H}$, $\{w_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ bounded weight-sequence

$$W \colon \mathcal{H} \to \mathcal{H}, \quad We_n = \begin{cases} w_{n-1} e_{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Graphs can be associated with these operators:



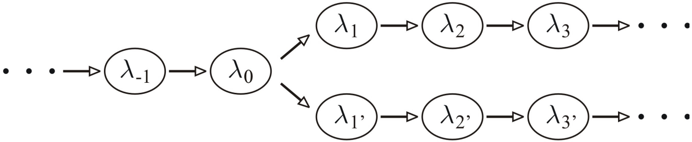
Some people write the weights on the edges.

A natural generalization: replace the previous graphs with directed trees and write some weights in the vertices which are not roots. $\longrightarrow$ This will naturally define a „tree-shift operator".

Z. J. Jablonski, I. B. Jung and **J. Stochel**, *Weighted Shifts on Directed Trees*, Memoirs of the American Mathematical Society, Number 1017, 2012.

(hyponormality, co-hyponormality, subnormality and hyperexpansivity – simpler examples than before)

An example



$We_n = \lambda_{n+1} e_{n+1}$, if $n \neq 0$

$We_{k'} = \lambda_{(k+1)'} e_{(k+1)'}$,

$We_0 = \lambda_1 e_1 + \lambda_{1'} e_{1'}$, if $n \neq 0$

$\mathcal{T} = (V, E)$ is a directed graph, $V \equiv$ vertices, $E \equiv$ (directed) edges $E \subseteq V \times V \setminus \{(v, v) \colon v \in V\}$. $\mathcal{T}$ is a directed tree if

(i) $\mathcal{T}$ is connected, i.e.: $\forall\, u, v \in V, u \neq v\ \exists$ $u = v_0, v_1, \ldots v_n = v \in V, n \in \mathbb{N}$ s. t. (v_{j-1}, v_j) or $(v_j, v_{j-1}) \in E$ for every $1 \leq j \leq n$.

(ii) For each vertex v there exists at most one other vertex u with the property that $(u, v) \in E$ (i.e. every vertex has at most one parent), and

(iii) $\mathcal{T}$ has no directed circuit, i.e.: $\nexists\ v_0, v_1, \ldots v_n \in V, n \in \mathbb{N}$ distinct vertices s. t. $(v_{j-1}, v_j) \in E\ \forall\, 1 \leq j \leq n$ and $(v_n, v_0) \in E$.

If $(u, v) \in E$, then v is a child of u, u is the parent of v.
In notations: $v \in \mathrm{Chi}(u)$, $\mathrm{par}_{\mathcal{T}}(v) = \mathrm{par}(v) = u$.
$\mathrm{Chi}_{\mathcal{T}}(u) = \mathrm{Chi}(u) \equiv$ the set of all children of u.
If v has a parent, then it is unique.

If u has no parent $\longrightarrow$ root. If there exists one, then it is unique
$u := \mathrm{root} = \mathrm{root}_{\mathcal{T}}$.

If $\mathrm{Chi}(u) = \emptyset \longrightarrow$ leaf. $\mathrm{Lea}(\mathcal{T}) \equiv$ the set of all leaves.

$\ell^2(V)$: complex Hilbert space of all square summable functions.
The natural inner product: $\langle f, g \rangle = \sum_{u \in V} f(u)\overline{g(u)}$.
$e_u(v) = \delta_{u,v}$ $(u \in V)$, $\{e_u\}_{u \in V}$ ONB.
$W \subseteq V \longrightarrow \ell^2(W) = \vee\{e_v : v \in W\}$ which is a subspace (closed linear manifold).

Bounded Tree-shift Operator

Let $\underline{\lambda} = \{\lambda_v \colon v \in V \setminus \{\mathrm{root}\}\} \subseteq \mathbb{C}$ be a set of weights
$\sup\left\{\sqrt{\sum_{v\in\mathrm{Chi}(u)} |\lambda_v|^2} \colon u \in V\right\} < \infty$

$$\mathbf{S}_{\underline{\lambda}} \colon \ell^2(\mathbf{V}) \to \ell^2(\mathbf{V}), \quad \mathbf{e_u} \mapsto \sum_{\mathbf{v}\in\mathrm{Chi}(\mathbf{u})} \lambda_{\mathbf{v}} \mathbf{e_v},$$

$$\|S_{\underline{\lambda}}\| = \sup\left\{\sqrt{\sum_{v\in\mathrm{Chi}(u)} |\lambda_v|^2} \colon u \in V\right\}$$

Asymptotic Behaviour of Hilbert Space Contractions

$\mathcal{H}$: complex Hilbert space, $\mathcal{B}(\mathcal{H})$: bounded linear operators on it.
Suppose $T \in \mathcal{B}(\mathcal{H})$ is a contraction: $\|T\| \leq 1$,
then the following limits in the strong operator topology exist:

$$A = A_T = \lim_{n\to\infty} T^{*n}T^n \quad \text{and} \quad A_* = A_{T^*} = \lim_{n\to\infty} T^nT^{*n}.$$

A: the asymptotic limit of T, A_*: the asymptotic limit of T^*.

$h \in \mathcal{H}$ is a stable vector for T, if $\lim_{n\to\infty} \|T^nh\| = 0$. The set of all stable vectors: $\mathcal{H}_0 = \mathcal{H}_0(T) = \mathcal{N}(A_T)$.
Easy to see that $\mathcal{H}_0$ is a hyperinvariant subspace of T and will be called the stable subspace of T.
(a subspace is hyperinvariant for T if it is invariant for every C commuting with T)

A Classification of Contractions (C. Foias and B. Sz.-Nagy)

If every vector is stable ($\iff A_T = 0$), then T is called stable or $C_{0\cdot}$ class contraction.
If $\mathcal{H}_0 = \{0\}$ ($\iff A_T$ is injective), then T is a $C_{1\cdot}$ class contraction.

If $T^* \in C_{i\cdot}(\mathcal{H})$ ($i = 0$ or 1), then T is of class $C_{\cdot i}$.
$C_{ij}(\mathcal{H}) = C_{i\cdot}(\mathcal{H}) \cap C_{\cdot j}(\mathcal{H})$.

Because $\mathcal{H}_0(T)$ or $\mathcal{H}_0(T^*)^\perp$ is hyperinvariant, if $T \notin C_{00} \cup C_{10} \cup C_{01} \cup C_{11}$, then it has a non-trivial hyperinvariant subspace.

$X \in \mathcal{B}(\mathcal{H}, \mathcal{R}(A_T)^-), Xh = A_T^{1/2}h.$
There exists a unique isometry $U \in \mathcal{B}(\mathcal{R}(A_T)^-)$ s. t.

$$XT = UX.$$

The pair (X, U) is a canonical realization of the so called isometric asymptote of T.

Application 1:

Theorem (C. Foias and B. Sz.-Nagy)

The contraction T is of class C_{11} iff T is quasi-similar to a unitary operator.

This solves HSP for C_{11} contractions.
The HSP remains open in case when $T \in C_{00}, C_{10}$ or C_{01}.

$T \sim U$ (quasi-similar), if $\exists\ X, Y \in \mathcal{B}(\mathcal{H})$ with dense range and trivial kernel s. t. $XT = UX$ and $YU = TY$.
(If they are similar, then $Y = X^{-1}$).

$h \in \mathcal{H}$ is a cyclic vector for $T \in \mathcal{B}(\mathcal{H})$

$$\vee\{T^n h \colon n \in \mathbb{Z}_+\} = \mathcal{H}.$$

Then T is cyclic operator.

Application 2:

Proposition

If U is not cyclic and $T \in C_{1\cdot}(\mathcal{H}) \implies T$ is also non-cyclic.
If U^* is cyclic and $T \in C_{1\cdot}(\mathcal{H}) \implies T^*$ is also cyclic

Assumption: all weights are **strictly positive**.
This can be assumed without loss of generality.

$$\mathrm{Br}(\mathcal{T}) = \sum_{\substack{u\in V \\ u\notin \mathrm{Lea}(\mathcal{T})}} (|\mathrm{Chi}(u)| - 1) \text{ is the branching index of } \mathcal{T}.$$

S^+ is the simple unilateral shift (all weights are 1) and S is the simple bilateral shift. They can be represented as multiplication operators by $\chi(z) = z$ on $H^2(\mathbb{D})$ and $L^2(\mathbb{T})$, respectively.

The isometric asymptote U of $S_{\underline{\lambda}}$

$\mathcal{T}' := (V', V' \times V' \cap E)$, where $\ell^2(V') = \mathcal{R}(A)^- = \mathcal{H}_0(S_{\underline{\lambda}})^{\perp}$. It can be shown that $\mathcal{T}'$ is always a directed tree.

Theorem

For a tree-shift contraction $S_{\underline{\lambda}} \notin C_{0\cdot}(\ell^2(V))$, the isometric asymptote $U \in \mathcal{B}(\ell^2(V'))$ is unitarily equivalent to:

(i) $\sum_{j=1}^{\mathrm{Br}(\mathcal{T}')+1} \oplus S^+$, *if $\mathcal{T}$ has a root,*

(ii) $\sum_{j=1}^{\mathrm{Br}(\mathcal{T}')} \oplus S^+$, *if $\mathcal{T}$ has no root and U has zero unitary part, i.e.:* $\sum_{v' \in \mathrm{Gen}_{\mathcal{T}'}(u')} \prod_{j=0}^{\infty} \beta^2_{\mathrm{par}^j(v')} = 0$ *for some (and then for every) $u' \in V'$,*

(iii) $S \oplus \sum_{j=1}^{\mathrm{Br}(\mathcal{T}')} \oplus S^+$, *if $\mathcal{T}$ has no root and U has a non-zero unitary part.*

The isometric asymptote U_* of $S_{\underline{\lambda}}^*$

Theorem

Suppose that $S_{\underline{\lambda}} \notin C_{\cdot 0}(\ell^2(V))$. Then $\mathcal{T}$ has no root and the isometry U_* is a simple unilateral shift, if $\mathrm{Chi}(\mathrm{Gen}(u)) = \emptyset$ for some $u \in V$, and a simple bilateral shift elsewhere.

With the two theorem above those tree-shift contractions that are similar to an isometry or a co-isometry can be characterized (with an easily computable formula). The proof of this uses the fact that a contraction T is similar to an isometry iff A_T is invertible.

Trivial non-cyclic cases

$h \in \mathcal{H}$ is a cyclic vector for $T \in \mathcal{B}(\mathcal{H})$

$$\vee\{T^n h \colon n \in \mathbb{Z}_+\} = \mathcal{H}.$$

Then T is cyclic operator.

Easy to see: If $\mathrm{co}-\dim(\mathcal{R}(T)) > 1 \implies T$ has no cyclic vectors.

$$\Downarrow$$

- if $\mathcal{T}$ has a root and $\mathrm{Br}(\mathcal{T}) > 0 \implies S_{\underline{\lambda}}$ has no cyclic vector,
- if $\mathcal{T}$ is rootless and $\mathrm{Br}(\mathcal{T}) > 1 \implies S_{\underline{\lambda}}$ has no cyclic vector.

$\mathrm{Br}(\mathcal{T}) = 0 \implies S_{\underline{\lambda}}$ is

- a weighted bilateral shift (no characterization for cyclicity is known) (for supercyclicity, hypercyclicity ... there are characterizations) (e.g.: articles of H. Salas and a monograph of A. L. Shields),
- a weighted unilateral shift (easy: always cyclic),
- a weighted backward shift (later),
- a cyclic nilpotent operator acting on a finite dimensional space.

S is a cyclic bilateral shift, but there are non-cyclic bilateral shifts as well (B. Beauzamy).

The pure tree-shift case is when $\mathrm{Br}(\mathcal{T}) = 1$ and $\mathcal{T}$ has no root.

The backward shift case

Theorem

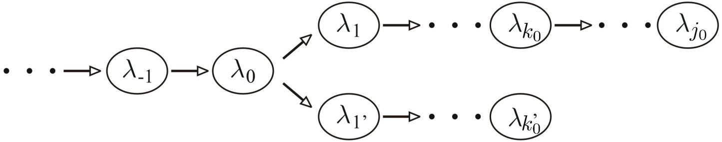
Suppose that B is a backward weighted shift of countable multiplicity. Then there is a cyclic vector f for B if and only if there is at most one zero weight.

The backward shift of multiplicity more than one is not a tree-shift operator!

In case, when the multiplicity is 1, this was obtained by **Z. Guang Hua** in 1984. This article was written in Chinese.

The proof for simple backward shift of countable multiplicity can be found in Halmos's Hilbert space problem book.

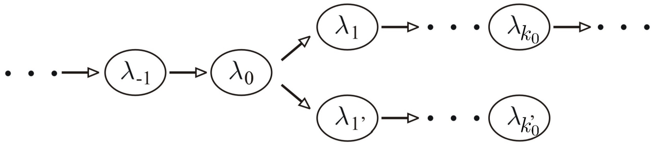
Two leaves



Theorem

If the directed tree $\mathcal{T}$ has no root, $\mathrm{Br}(\mathcal{T}) = 1$ and have 2 leaves, then every bounded tree-shift operator on it is cyclic.

One leaf



Theorem

Suppose $\mathcal{T}$ has a unique leaf. A tree-shift operator $S_{\underline{\lambda}}$ on $\mathcal{T}$ is cyclic if and only if the bilateral shift W with weights $\{\lambda_n\}_{n=-\infty}^{\infty}$ is cyclic. In particular, if $S_{\underline{\lambda}} \notin C_{\cdot 0}(\ell^2(V))$, then $S_{\underline{\lambda}}$ is cyclic.

No leaves

Proposition

The operator $S \oplus S^+$ has no cyclic vector.

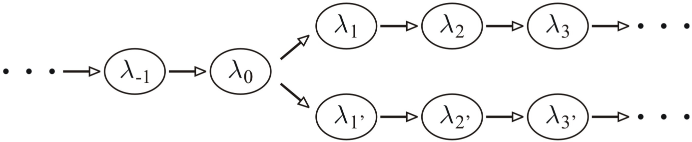
Theorem

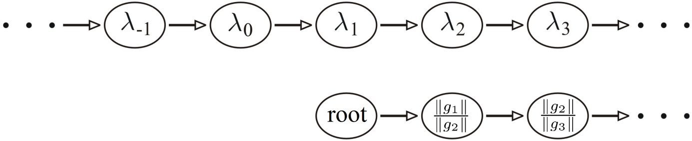
Suppose that $\mathcal{T}$ is rootless and $\mathrm{Br}(\mathcal{T}) = 1$. If the tree-shift contraction $S_{\underline{\lambda}}$ is of class $C_1.$, then it has no cyclic vector.

Proof. The isometric asymptote of $S_{\underline{\lambda}}$ is unitarily equivalent to $S \oplus S^+$. ∎

On this $\mathcal{T}$ the contrary may also happen.

Similarity to orthogonal sum of bi- and unilateral shifts





$$g_k = \prod_{j=1}^{k} \frac{1}{\lambda_j} \cdot e_k - \prod_{j=1}^{k} \frac{1}{\lambda_{j'}} \cdot e_{k'} \qquad (k \in \mathbb{N}) \ .$$

Theorem

If $\left\{ \prod_{j=1}^{k} \frac{\lambda_{j'}}{\lambda_j} : k \in \mathbb{N} \right\}$ *is bounded, then* $S_{\underline{\lambda}}$ *is similar to the previous orthogonal sum.*

Corollary

If $S_{\underline{\lambda}} \notin C_{0\cdot}(\ell^2(V))$*, then the similarity holds.*

Theorem

There is a tree-shift operator on the previous directed tree which is cyclic.

Cyclicity of the adjoint

$$S_k^+ := \underbrace{S^+ \oplus \cdots \oplus S^+}_{k \text{ times}} \ (k \in \mathbb{N}),$$

$$S_{\aleph_0}^+ := \underbrace{S^+ \oplus S^+ \oplus \ldots}_{\aleph_0 \text{ many}}.$$

Theorem (L. Kérchy, GyPG)

The operator $S \oplus (S_k^+)^*$ is cyclic for every $k \in \mathbb{N}$.

Question

Is the operator $S \oplus (S_{\aleph_0}^+)^*$ cyclic?

Theorem

The followings are valid:

(i) *If $\mathcal{T}$ has a root and the tree-shift contraction $S_{\underline{\lambda}}$ on it is of class $C_{1\cdot}$, then $S_{\underline{\lambda}}^*$ is cyclic.*

(ii) *If $\mathcal{T}$ is rootless, $\mathrm{Br}(\mathcal{T}) < \infty$ and the tree-shift contraction $S_{\underline{\lambda}}$ on it is of class $C_{1\cdot}$, then $S_{\underline{\lambda}}^*$ is cyclic.*

Thank You for Your Attention!