Parabolic Non-automorphism induced Toeplitz-Composition C*-Algebras with Piece-wise Quasi-continuous Symbols

Uğur Gül

Hacettepe University,
Ankara

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Notation

- $\mathbb{C}$-the complex plane
- $\mathbb{D}$-the unit disc
- $\mathbb{H}$-the upper half-plane
- $\mathfrak{C}(z) = \frac{z-i}{z+i}$-the Cayley transform
- $\mathcal{F}$-the Fourier transform
- $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$-the space of quasi-continuous functions
- PC-the algebra of bounded piece-wise continuous functions
- PQC-the C*-algebra generated by bounded piece-wise continuous functions and quasi-continuous functions

Main Result

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Theorem

Let $a \in PC(\mathbb{T})$ and $\eta \in QC(\mathbb{T})$ with $\Im(\eta(z)) > \epsilon > 0$ for all $z \in \mathbb{D}$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ of the following form

$$\varphi(z) = \frac{2iz + \eta(z)(1-z)}{2i + \eta(z)(1-z)}$$

then for $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ and $T_a : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ we have for $s \in [0,1]$ and $t \in [0,\infty)$ running arbitrarily,

- $\sigma_e(T_a C_\varphi) = \overline{\{(sa(1^-) + (1-s)a(1^+))e^{izt} : z \in \mathcal{C}_1(\eta)\}}$
- $\sigma_e(T_a + C_\varphi) = \overline{\{(sa(\lambda^-) + (1-s)a(\lambda^+)) + e^{izt} : z \in \mathcal{C}_1(\eta), \lambda \in \mathbb{T}\}}$

where $\mathcal{C}_\lambda(\eta)$ of $\eta \in H^\infty$ is the set of cluster points of η at λ, $a(\lambda^+) = \lim_{\theta \to 0^+} a(\lambda e^{i\theta})$ and $a(\lambda^-) = \lim_{\theta \to 0^-} a(\lambda e^{i\theta})$.

The Hardy Space H^2

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$$< f, g > = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta,$$

in $L^2(\mathbb{T})$

Toeplitz Operators

Let $P : L^2(\mathbb{T}) \to H^2$ be the orthogonal projection of L^2 onto H^2 and $\phi \in L^\infty(\mathbb{T})$ then the Toeplitz operator $T_\phi : H^2 \to H^2$ is defined as

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$$T_\phi = PM_\phi \mid_{H^2}$$

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Note that C_φ is linear and bounded on H^2 by Littlewood subordination principle.

Paley-Wiener Theorem and Fourier Multipliers

The Hardy space of the upper half-plane $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$ is the closure of the linear span $< \{\frac{1}{x+i}\left(\frac{x-i}{x+i}\right)^n\}_{n=0}^{\infty} >$ with respect to the inner product

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$$D_\vartheta = \Phi^{-1} \circ \mathcal{F}^{-1} \circ M_\vartheta \circ \mathcal{F} \circ \Phi$$

where $M_\vartheta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the multiplication operator $(M_\vartheta f)(t) = \vartheta(t)f(t)$.

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$$T_{\frac{2i(1-z)}{2i+a(1-z)}} C_{\varphi_a} = D_{e^{iat}} \tag{1}$$

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$$\mathcal{T}(A) = C^*(\{T_\phi : \phi \in A\})$$

generated by Toeplitz operators with symbols in A.

C*-algebras with Quasi-continuous Symbols

Douglas has extended the above results of Coburn from continuous symbols to Quasi-continuous symbols namely the semi-commutators are also compact when the symbols are taken from $QC(\mathbb{T})$ and S also extends to be an isometric isomorphism when $C(\mathbb{T})$ is replaced by $QC(\mathbb{T})$.

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C*-algebras with Piece-wise Continuous Symbols

Gohberg and Krupnik have analyzed $\mathcal{T}(PC(\mathbb{T}))$, the C*-algebras of Toeplitz Operators with piece-wise continuous symbols. They showed that the commutators are compact however the semi-commutators are not. Hence $\mathcal{T}(PC(\mathbb{T}))/K(H^2)$ is still a commutative C*-algebra however it is not isometrically isomorphic to $PC(\mathbb{T})$.

C*-algebras with Piece-wise Quasi-continuous Symbols

Based on the above results Sarason observed that the C*-algebra $\mathcal{T}(PQC(\mathbb{T}))/K(H^2)$ generated by Toeplitz operators with piece-wise quasi-continuous symbols is also commutative and analyzed its maximal ideal space.

The Toeplitz-Composition C*-algebra

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$$\Psi(PQC(\mathbb{T}), C([0,\infty])) = C^*(\mathcal{T}(PQC(\mathbb{T})) \cup F(C([0,\infty])))$$

the C*-algebra generated by Toeplitz operators with piece-wise continuous symbols and continuous Fourier multipliers.

The Toeplitz-Composition C*-algebra modulo compact operators

Compactness of Commutators

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Hence the C*-algebra $\Psi(PQC(\mathbb{T}), C([0,\infty]))/K(H^2)$ is commutative

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Our C*-algebra is commutative, hence it is of interest to characterize its maximal ideal space. The main tool in doing this is the following theorem due to Power:

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Theorem

Let C_1, C_2 and C_3 be three C-subalgebras of $B(H)$ with identity, where H is a separable Hilbert space, such that $M(C_i) \neq \emptyset$, where $M(C_i)$ is the space of multiplicative linear functionals of C_i, $i = 1, 2, 3$ and let C be the C*-algebra that they generate. Then we have $M(C) = P(C_1, C_2, C_3) \subset M(C_1) \times M(C_2) \times M(C_3)$, where $P(C_1, C_2, C_3)$ is defined to be the set of points $(x_1, x_2, x_3) \in M(C_1) \times M(C_2) \times M(C_3)$ satisfying the condition:*

Given $0 \leq a_1 \leq 1$, $0 \leq a_2 \leq 1$, $0 \leq a_3 \leq 1$ $a_i \in C_i, i = 1, 2, 3$

$$x_i(a_i) = 1 \quad with \quad i = 1, 2, 3 \quad \Rightarrow \quad \|a_1 a_2 a_3\| = 1.$$

To apply Power's theorem we take $C_1 = \mathcal{T}(PC(\mathbb{T}))$, $C_2 = \mathcal{T}(QC(\mathbb{T}))$ and $C_3 = F(C([0,\infty]))$ and we obtain the following result

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Theorem

Let $\Psi = \Psi(PQC(\mathbb{T}), C([0,\infty])$. Then $\Psi / K(H^2)$ is a commutative C^*-algebra and its maximal ideal space is characterized as

$$M(\Psi) \cong (M_1(\mathcal{T}(PC)) \times M_1(QC) \times [0,\infty]) \cup ([\cup_{\lambda \in \mathbb{T}}(M_\lambda(\mathcal{T}(PC)) \times M_\lambda(QC))] \times \{\infty\})$$

where
$M_\lambda(\mathcal{T}(PC)) = \{x \in M(\mathcal{T}(PC)) : x \mid_{C(\mathbb{T})} = \delta_\lambda, \delta_\lambda(T_f) = f(\lambda)\}$ *and* $M_\lambda(QC) = \{x \in M(QC) : x \mid_{C(\mathbb{T})} = \delta_\lambda, \delta_\lambda(T_f) = f(\lambda)\}$ *are the fibers of* $M(\mathcal{T}(PC))$ *and* $M(QC)$ *at* λ *respectively.*

An Application

Essential Spectra of Linear Combinations of Composition Operators and Toeplitz Operators

We apply the above result in determining the essential spectra of certain weighted composition operators using the following result due to G.:If $\varphi : \mathbb{D} \to \mathbb{D}$ is of the following form

$$\varphi(z) = \frac{2iz + \eta(z)(1-z)}{2i + \eta(z)(1-z)}$$

where $\eta \in H^{\infty}(\mathbb{D})$ with $\Im(\eta(z)) > \epsilon > 0$ for all $z \in \mathbb{D}$, then $\exists \alpha \in \mathbb{R}^{+}$ such that

$$C_{\varphi} = T_{\frac{2i+\eta(z)(1-z)}{2i}} \sum_{n=0}^{\infty} T_{(i\alpha - \eta(z))^{n}} D_{\vartheta_{n}},$$

where $\vartheta_n(t) = \frac{(-it)^n e^{-\alpha t}}{n!}$.

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