Function Spaces on Varieties

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me -UF

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$$\mathcal{Z}_p =: \{(z, w) \in \mathbb{C}^2 : p(z, w) = 0\}$$
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for all \( (z, w) \in \mathcal{Z}_p, |z| = 1 \) if and only if \( |w| = 1 \). \hspace{1cm} (DV)
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- $p(z, w) = z - w$ (boring)
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- \( p(z, w) = z^2 - w^2 = (z - w)(z + w) \) (mostly boring)
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- \( p(z, w) = z^2 - w^2 = (z - w)(z + w) \) (mostly boring)
- \( p(z, w) = z^3 - w^2 \) (not boring)
The zero variety in the last example

\[ p(z, w) = z^3 - w^2 \]

is called the **Neil parabola**. Since

\[ \text{grad}_C p = \left( \frac{\partial p}{\partial z}, \frac{\partial p}{\partial w} \right) = (3z^2, -2w), \]

the origin \((0, 0)\) is a **singular point** (in this case, a **cusp**).
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the origin \((0, 0)\) is a **singular point** (in this case, a **cusp**). Mostly we are interested in analytic functions on

\[ \mathcal{V} = Z_p \cap \mathbb{D}^2. \]

Notice \(\partial \mathcal{V} \subset \partial \mathbb{D} \times \partial \mathbb{D} \subset \partial (\mathbb{D} \times \mathbb{D}).\)
More examples:

\[ z^m = \prod_{j=1}^{n} \frac{a_j - w}{1 - a_j^*w}, \text{ with } a_j \text{ distinct, nonzero} \]

The set

\[ \mathcal{V} = \{(z, w) \in \mathbb{D}^2 : z^m = B_n(z)\} \]

is a smooth distinguished variety; in fact it is a finite Riemann surface.
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defines a finite Riemann surface of genus

\[ g = \frac{(m - 1)(n - 1) - (k - 1)}{2} \]

with \( k = \text{gcd}(m, n) \) disks removed.
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- \( m = n = 2 \): annulus, every annulus arises this way (Bell)
- every finitely connected planar domain is a variety, but not smooth if 2 or more holes (Rudin, Fedorov)
Definition

Let $\mathcal{V}$ be a distinguished variety. A function $f : \mathcal{V} \to \mathbb{C}$ is holomorphic at $(z, w) \in \mathcal{V}$ if there exist:
- a neighborhood $\Omega$ of $(z, w)$ in $\mathbb{C}^2$, and
- a holomorphic function $F : \Omega \to \mathbb{C}$

such that

$$F|_{\mathcal{V} \cap \Omega} = f|_{\mathcal{V} \cap \Omega}.$$
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FACT (H. Cartan): If $f$ is holomorphic on $\mathcal{V}$ then there exists an $F$ holomorphic on $\mathbb{D}^2$ such that $F|_{\mathcal{V}} = f$. 

Example:

What do the holomorphic functions on the Neil parabola $N$ look like?
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**FACT (H. Cartan):** If $f$ is holomorphic on $\mathcal{V}$ then there exists an $F$ holomorphic on $\mathbb{D}^2$ such that $F|_{\mathcal{V}} = f$.

**Example:** what do the holomorphic functions on the Neil parabola $\mathcal{N}$ look like?
The Neil parabola

\[ \mathcal{N} = \{(z, w) \in \mathbb{D}^2 : z^3 = w^2\} \]

is parameterized by the disk: the map

\[ \psi : t \to (t^2, t^3) \]

is a holomorphic bijection of \( \mathbb{D} \) with \( \mathcal{N} \).

\[ f \to f \circ \psi \] is an isomorphism of \( \text{Hol}(\mathcal{N}) \) with

\[ \{ f \in \text{Hol}(\mathbb{D}) : f'(0) = 0 \}. \]
\[ \mathcal{N} = \{(z, w) \in \mathbb{D}^2 : z^3 = w^2 \} \]

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Take \( f \) holomorphic on \( \mathcal{N} \), extend to \( \mathbb{D}^2 \) (Cartan): then

\[
f(\psi(t)) = \sum_{m,n=0}^{\infty} a_{mn}(t^2)^m(t^3)^n = \sum_{k \neq 0} b_k t^k
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Conversely for \( k \neq 1 \) write \( t^k = (t^2)^m(t^3)^n \), then

\[ \sum_{k \neq 0} b_k t^k = \sum_{m,n} b_{2m+3n}(t^2)^m(t^3)^n \]
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**Conclusion:**

\[ \text{Hol}(\mathcal{N}) \cong \{ f \in \text{Hol}(\mathbb{D}) : f'(0) = 0 \} \]
This was a special case of a general fact:

Theorem (Agler-McCarthy, 2007)

Let $\mathcal{V} \subset \mathbb{D}^2$ be a distinguished variety. Then there exist:

- a finite Riemann surface $R$,
- a holomorphic map $\psi : R \to \mathcal{V}$, and
- a finite codimension subalgebra $A \subset \text{Hol}(R)$

such that

$$f \mapsto f \circ \psi$$

is an isomorphism of $\text{Hol}(\mathcal{V})$ with $A$. 
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**Theorem (Agler-McCarthy, 2007)**

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The map $\psi$ has the form

$$t \to (\psi_1(t), \psi_2(t))$$

with $|\psi_1| = |\psi_2| = 1$ on $\partial R$, and $p(\psi_1, \psi_2) = 0$. In other words, an algebraic pair of inner functions on $R$. 

Determinantal representations:

**Theorem (Agler-McCarthy 2005, Knese 2009)**

If $\Phi$ is a rational matrix inner function, then

$$\det(wI - \Phi(z)) = 0$$

defines a distinguished variety; conversely every distinguished variety may be put in this form.

**Example:**

$\Phi(z) = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}$

then

$$\det(wI - \Phi(z)) = \det(w - z - z^2w) = w^2 - z^3,$$

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Example:

$$\Phi(z) = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}$$

then

$$\det(wI_2 - \Phi(z)) = \det \begin{pmatrix} w & -z \\ -z^2 & w \end{pmatrix} = w^2 - z^3,$$

the Neil parabola.
Idea of proof:

Let $V = Z_q \cap D^2$ and choose a nice measure $\mu$ on $\partial V$ (e.g. push down harmonic measure from $\partial R$).

Operators $M_z, M_w \in B(H^2(\mu))$ are commuting isometries with $q(M_z, M_w) = 0$. The pair has a Sz.-Nagy–Foias model $M_z \sim S \otimes I_m, M_w \sim \Phi(S)$ for a matrix inner function $\Phi$.

Or we could take $M_z \sim \Psi(S), M_w \sim S \otimes I_n$.

Then $\det(zI_m - \Phi(w)) = \det(wI_n - \Psi(z)) = 0$ defines $V$. 

Mike Jury  
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Then

$$\det(zI_m - \Phi(w)) = \det(wI_n - \Psi(z)) = 0$$

defines $\mathcal{V}$. 
Application: Nevanlinna-Pick interpolation on $V$

General question: given points $s_1, ..., s_n$ in a (domain, surface, variety) $\Omega$, and points $t_1, ..., t_n$ in the unit disk $D$, when does there exist a holomorphic function $f: \Omega \to D$ with $f(s_i) = t_i$, all $i = 1, ..., n$?

Theorem (Pick(1916), Nevanlinna(1919),...)

For $\Omega = D$, the interpolation problem has a solution if and only if the $n \times n$ matrix $1 - t_i t^*_j$ is positive semidefinite.
**Application:** Nevanlinna-Pick interpolation on \( V \)

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\[ s_1, \ldots, s_n \]

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**Theorem (Pick(1916), Nevanlinna(1919),...)**

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\frac{1 - t_i t_j^*}{1 - s_i s_j^*}
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\frac{1 - t_i t_j^*}{1 - s_i s_j^*} = (1 - t_i t_j^*) k(s_i, s_j)
$$

is positive semidefinite.
Determinantal representation $\Rightarrow$ reproducing kernels:

For each matrix inner function giving a determinantal representation $V = \{ \det(wI - \Phi(z)) = 0 \}$, there is an analytic (on $V$) family of "eigenvectors" $Q(z, w)\Phi(z) = wQ(z, w)$ for all $(z, w) \in V$.

Example: $\begin{bmatrix} w & z \\ z & w \end{bmatrix} = \begin{bmatrix} z & w \\ w & z \end{bmatrix} = \begin{bmatrix} z^3 & wz \\ wz & z^2 \end{bmatrix}$ since $z^3 = w^2$ on $N$.
Determinantal representation $\iff$ reproducing kernels:

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for all $(z, w) \in \mathcal{V}$. 

Example:

$$(w \ z)^{(0 \ z \ z^2 \ 0)} = (z^3 wz)^{(0 \ z \ z \ z^2)} = w(Q(z, w))$$
since $z^3 = w^2$ on $\mathcal{V}$. 

Determinantal representation $\implies$ reproducing kernels:

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for all $(z, w) \in V$.

**Example:**

$$\begin{pmatrix} w & z \\ z^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^3 & wz \end{pmatrix} = w \begin{pmatrix} w & z \end{pmatrix}$$

since $z^3 = w^2$ on $N$. 
In addition to

\[ Q(z, w)\Phi(z) = wQ(z, w) \]

we have \( P(z, w) \) so that for the “companion” matrix inner function \( \Psi \),

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we have \( P(z, w) \) so that for the "companion" matrix inner function \( \Psi \),
\[ P(z, w) \Psi(w) = zP(z, w) \]
For \((z, w) \in V, (\zeta, \eta) \in V\), we have the identity
\[ \frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta} = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\eta} \]
Conversely, if \( P, Q \) are vector polynomials satisfying the identity, the come from a determinantal representation.
\[
\frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta} = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\eta}
\]

Call the above expression \( K((z, w), (\zeta, \eta)) \). Then
\[
\frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta} = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\overline{\eta}}
\]

Call the above expression \( K((z, w), (\zeta, \eta)) \). Then

**Theorem (Knese, McCullough, J.)**

Let \( V \) be an irreducible distinguished variety. There exists \( f \in H^\infty(V) \) with

\[
f(z_i, w_i) = t_i \quad \text{and} \quad \|f\|_\infty \leq 1
\]

if and only if the matrices

\[
(1 - t_it_j^*)K((z_i, w_i), (z_j, w_j))
\]

are all positive semidefinite, over all choices of determinantal representation.
For the Neil parabola, we had

\[ H^\infty(\mathcal{N}) \cong \{ f \in H^\infty(\mathbb{D}) : f'(0) = 0 \}. \]

Interpolation theorem recovers result of Davidson-Paulsen-Raghupathi-Singh
Ingredients of proof:

\[
\text{Step 1)} \text{ The kernels } Q(z, w) Q(\zeta, \eta) = P(z, w) P(\zeta, \eta) (1)
\]

over all determinantal representations form a complete Nevanlinna-Pick family: fix one such kernel \( k \) with associated \( \text{RKHS} H_2(k) \). Fix \((z_0, w_0) \in V\).

The kernel for the subspace \(
\{ f \in H_2(k) : f(z_0, w_0) = 0 \}
\)

has the form

\[
1 - z_0 z^* (1 - w_0 \eta^*) \tilde{K}((z, w), (\zeta, \eta)) (1 - z \zeta^*) (1 - w \eta^*)
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Ingredients of proof:

Step 1) The kernels

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\frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta} = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\bar{\eta}}
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(1)

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\[
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has the form

\[
\frac{1}{(1 - zz_0^*)(1 - ww_0^*)} \tilde{K}((z, w), (\zeta, \eta)) \frac{1}{(1 - z_0\zeta^*)(1 - w_0\eta^*)}
\]
Step 2) Define a new norm on polynomials by

$$\|p\|_{\mathcal{K}} = \sup_{\mathcal{K}} \{ \| M_p \|_{B(H^2(K))} \}$$

Now by Step 1, interpolation problem is solved for $H^\infty_{\mathcal{K}}$ norm by a general interpolation theorem for kernel families (Knese-McCullough-J)
Step 3) Finally, prove the $\mathcal{K}$ norm equals the supremum norm on $\mathcal{V}$—this follows from

**Lemma**

If $f \in H^\infty(\mathcal{V})$ and $\|f\|_\infty \leq 1$, there is a sequence of polynomials $p_n \in \mathbb{C}[z, w]$ such that

- $|p_n| \leq 1$ on $\mathcal{V}$, and
- $p_n \to f$ locally uniformly on $\mathcal{V}$
Dilations & spectral sets:

Let $q(z, w)$ define a distinguished variety $\mathcal{V}$. 

Necessary: for all polynomials $p$,

$$
\|p(S, T)\| \leq \|p\|_{V, \infty}
$$
Dilations & spectral sets:

Let \( q(z, w) \) define a distinguished variety \( \mathcal{V} \).

**Question:** given commuting contractive operators \( S, T \in B(H) \) with \( q(S, T) = 0 \), when do there exist commuting unitary operators \( U, V \in B(K) \) with \( q(U, V) = 0 \) and

\[
S^m T^n = P_H U^m V^n|_H
\]

Necessary: for all polynomials \( p \),

\[
\|p(S, T)\| \leq \|p\|_{\mathcal{V}, \infty}
\]
A homomorphism

\[ \pi : A(V) \to B(H) \]

is contractive if

\[ \| \pi(f) \|_{B(H)} \leq \| f \|_{\infty} \]

for all \( f \in A(V) \),
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\[ \pi : A(V) \to B(H) \]
is contractive if
\[ \|\pi(f)\|_{B(H)} \leq \|f\|_{\infty} \]
for all \( f \in A(V) \),
and completely contractive if
\[ \|[\pi(f_{ij})]\|_{B(H)} \leq \|[f_{ij}]\|_{\infty} \]
for all matrices \( f \in M_n(A(V)) \).
A homomorphism
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is **contractive** if
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for all matrices \( f \in M_n(A(V)) \).

Arveson: \( S, T \) dilate if and only if the contractive homomorphism
\[ \pi(p) = p(S, T) \]
is completely contractive.
Say rational dilation holds on $\mathcal{V}$ if every contractive $\pi$ is completely contractive.
Say **rational dilation holds on** \( \mathcal{V} \) if every contractive \( \pi \) is completely contractive.

Rational dilation
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Rational dilation holds on the disk (Sz.-Nagy dilation)
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Rational dilation holds on the disk (Sz.-Nagy dilation)
...holds on the annulus (Agler)
...fails on a two-holed planar domain (Dritschel-McCullough, Agler-Harman-Raphael)
Theorem (Dritschel, McCullough, J.)

Rational dilation fails on the Neil parabola \( \mathcal{N} \).

In fact: there is a representation \( \pi: \mathcal{A}(\mathcal{N}) \to M_{12} \times 12(\mathbb{C}) \) contractive but not 2-contractive.

As before identify \( \mathcal{A}(\mathcal{N}) \) with subalgebra of \( \mathcal{A}(\mathcal{D}) \):

\[
\mathcal{A}(\mathcal{N}) = \{ f \in \mathcal{A}(\mathcal{D}) : f'(0) = 0 \}.
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Convexity approach (Agler):

Let

\[ \mathcal{P} = \left\{ \frac{1 + f}{1 - f} : f \in H^\infty(N), f(0) = 1 \right\} \]

The set \( \mathcal{P} \) is compact and convex, let

\[ \mathcal{E} = \text{extreme points of } \mathcal{P} \]
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\[ \mathcal{E} = \text{extreme points of } P \]

For each \( f \in P \) we have a Choquet integral

\[ \frac{1 + f}{1 - f} = \int_{\mathcal{E}} \frac{1 + \phi_t}{1 - \phi_t} \, d\mu(t) \]

The functions

\[ \{ \phi : \frac{1 + \phi}{1 - \phi} \in \mathcal{E} \} \]

are called test functions for \( H^\infty(\mathcal{N}) \).
Representing the unit ball of $H^\infty(\mathcal{N})$: 

\begin{equation}
1 - f(z) f(w) = \int_{E} 1 - \phi_t(z) \phi_t(w) d\mu_{zw}(t)
\end{equation}

where $\mu$ is a positive measure valued kernel on $D \times D$. To proceed we need this more explicitly...
Representing the unit ball of $H^\infty(\mathcal{N})$:

Rearranging the Choquet integral we have

$$1 - f(z)f(w)^* = \int_\mathcal{E} 1 - \phi_t(z)\phi_t(w)^* d\mu_{zw}(t)$$

where $\mu$ is a positive measure valued kernel on $\mathbb{D} \times \mathbb{D}$. To proceed we need this more explicitly...
\( \mathbb{D}^* = \text{one-point compactification of } \mathbb{D} \)

\[ \psi_\lambda(z) = z^2 \frac{\lambda - z}{1 - \lambda z}, \quad \psi_\ast(z) = z^2 \text{ (test functions)} \]
\( D^* = \) one-point compactification of \( \mathbb{D} \)

\[ \psi_\lambda(z) = z^2 \frac{\lambda - z}{1 - \lambda z}, \quad \psi^*(z) = z^2 \text{ (test functions)} \]

**Theorem (Pickering)**

*If \( f \) lies in the unit ball of \( A(\mathbb{N}) \) and is smooth across the boundary then*

\[
1 - f(z)f(w)^* = \int_{D^*} (1 - \psi_t(z)\psi_t(w)^*) \, d\mu_{z,w}(t)
\]

*where \( \mu \) is a positive \( M(D^*) \)-valued kernel.*
The test functions can be pushed back down to $\mathcal{N}$, we get

$$
\phi_\lambda(z, w) = z \frac{\lambda z - w}{z - \lambda^* w}, \phi_*(z) = z
$$

Pickering also shows no (closed) subcollection of test functions suffices.
The test functions can be pushed back down to $\mathcal{N}$, we get

$$\phi_\lambda(z, w) = z \frac{\lambda z - w}{z - \lambda^* w}, \phi^*(z) = z$$

Pickering also shows no (closed) subcollection of test functions suffices. **Corollary:** a pair of commuting, contractive, invertible matrices $X, Y$ with $X^3 = Y^2$ give a contractive representation of $\mathcal{A}(\mathcal{N})$ if and only if

$$X(\lambda X - Y)(X - \lambda^* Y)^{-1}$$

is contractive for all $\lambda \in \mathbb{D}$.
Loosely, moving to matrix valued $F$, if $F$ every $n \times n$ matrix function also has a representation

$$1 - F(z)F(w)^* = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*) d\mu_{z,w}(t)$$

then one can pass (nice) representations inside the integral to conclude contractive implies completely contractive.
Conversely, let $\mathcal{F}$ be a finite set and form the closed, convex cone

$$C_{\mathcal{F}} = \left\{ H(z, w) = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*) \, d\mu_{z,w}(t) \right\}$$

where $\mu_{z,w}$ are matrix-valued measures. If for some $F$ we have

$$I - F(z)F(w)^* \notin C_{\mathcal{F}}$$

then we can separate $I - F(z)F(w)^*$ from $C_{\mathcal{F}}$ with a positive functional, apply GNS to get a representation that is contractive but NOT completely contractive.
Key step: if $F$ is a matrix inner function in $M_2 \otimes \mathcal{A}(\mathcal{N})$, an integral representation for $F$ imposes constraints on its zeroes...
\[ F(z) = z^2 \Phi(z), \quad \Phi \text{ rational, inner, degree 2}, \]

**Theorem**

*If* \( F \) *is representable as*

\[
I - F(z)F(w)^* = \int_{D^*} (1 - \psi_t(z)\psi_t(w)^*) \, d\mu_{z,w}(t)
\]

*for* \( z, w \) *in a large finite set* \( \mathcal{S} \), *then either*

\[
\Phi \simeq \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & \phi_1\phi_2 \end{pmatrix}
\]