Polynomials with no zeros on a face of the bidisk

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BIDISK

$$\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$$

UNIT DISK

FACE OF BIDISK

$$\mathbb{T} \times \mathbb{D}$$

UNIT CIRCLE

- POLYNOMIALS WITH NO ZEROS ON $\mathbb{D}^2$ (PRETTY) WELL UNDERSTOOD
- POLYNOMIALS WITH NO ZEROS ON $\mathbb{T} \times \mathbb{D}$ BEHAVE SURPRISINGLY LIKE THE ABOVE POLYNOMIALS
- WHAT CAN BE SAID ABOUT POLYNOMIALS WITH NO ZEROS ON $\mathbb{T}^2$?

MOTIVATION: SCHUR-COHN THM.

LET $p \in \mathbb{C}[z]$ WITH $\deg p = d$

Let $\overleftarrow{p}(z) = z^d\,\overline{p(1/z)}$.

SUPPOSE $\gcd(p, \overleftarrow{p}) = 1$.

HOW CAN WE COUNT THE ZEROS OF p INSIDE AND OUTSIDE $\mathbb{T}$?

$$p(z) = \sum_{k=0}^{d} p_k z^k$$

$$A = \begin{pmatrix} p_0 & & & 0 \\ p_1 & p_0 & & \\ \vdots & & \ddots & \\ p_{d-1} & \cdots & & p_0 \end{pmatrix} \begin{pmatrix} \; \end{pmatrix}^* - \begin{pmatrix} \bar{p}_d & & & 0 \\ \bar{p}_{d-1} & \bar{p}_d & & \\ \vdots & & \ddots & \\ \bar{p}_1 & \cdots & & \bar{p}_d \end{pmatrix} \begin{pmatrix} \; \end{pmatrix}^*$$

$n_1 =$ # zeros of $\tilde{p}$ in $\mathbb{D}$ = # POS. EIGENVALUES OF A

$n_2 =$ # ZEROS OF p IN $\mathbb{D}$ = # NEG EIGENVALUES OF A

EQUIVALENT TO A SUM OF SQUARES FORMULA:

$$\frac{|p(z)|^2 - |\tilde{p}(z)|^2}{1 - |z|^2} = (1, \bar{z}, \dots, \bar{z}^{d-1}) A \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{d-1} \end{pmatrix}$$

$$= \sum_{k=1}^{n_1} |a_k(z)|^2 - \sum_{k=1}^{n_2} |b_k(z)|^2$$

WHAT HAPPENS IN 2 VARIABLES?

QUICK ASIDE: ZEROS ON $\mathbb{T}$

OR PAIRS OF "REFLECTED" ZEROS $z, 1/\bar{z}$ CAUSE PROBLEMS.

QUESTION (A. SOKAL)

FIND A SELF-ADJOINT MATRIX DEPENDING ALGEBRAICALLY ON THE COEFFICIENTS OF $p \in \mathbb{C}[z]$ WHOSE SIGNATURE EXACTLY COUNTS ZEROS IN $\mathbb{D}$, $\mathbb{T}$, $\mathbb{E}$

THM: Suppose $p \in \mathbb{C}[z,w]$ has no zeros on $\overline{\mathbb{T}} \times \mathbb{D}$ and no factors in common with $\overleftarrow{p}(z,w) = z^n w^m \overline{p\left(\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right)}$ $\left(\begin{matrix}\deg p \\ = (n,m)\end{matrix}\right)$.

THEN, $\exists$ $A \in \mathbb{C}^{n_1}[z,w]$, $B \in \mathbb{C}^{n_2}[z,w]$, $C \in \mathbb{C}^{m}[z,w]$ such that

$$|p|^2 - |\overleftarrow{p}|^2 = (1-|z|^2)\left(|A(z,w)|^2 - |B(z,w)|^2\right) + (1-|w|^2)\,|C(z,w)|^2$$

$n_2 = \#$ ZEROS OF $p(z,0)$ IN $\mathbb{D}$

$n_1 = n - n_2$

- SIMULTANEOUSLY GENERALIZES SCHUR-COHN RESULT AND COLE-WERMER/GERONIMO-WOERDEMAN FORMULA FOR POLYNOMIALS WITH NO ZEROS ON $\mathbb{D}^2$

- ULTIMATELY, WE'D LIKE TO GENERALIZE TO CASE OF $\gcd(p, \overleftarrow{p}) = 1$.

TWO PROOFS

① GOES THROUGH CHARACTERIZATION OF FEJÉR-RIESZ FACTORIZATIONS (ONLY WORKS FOR NO ZEROS ON $\mathbb{T} \times \mathbb{D}$)

② USES A METHOD DUE TO A. KUMMERT TO REDUCE TO ONE VARIABLE RESULTS.

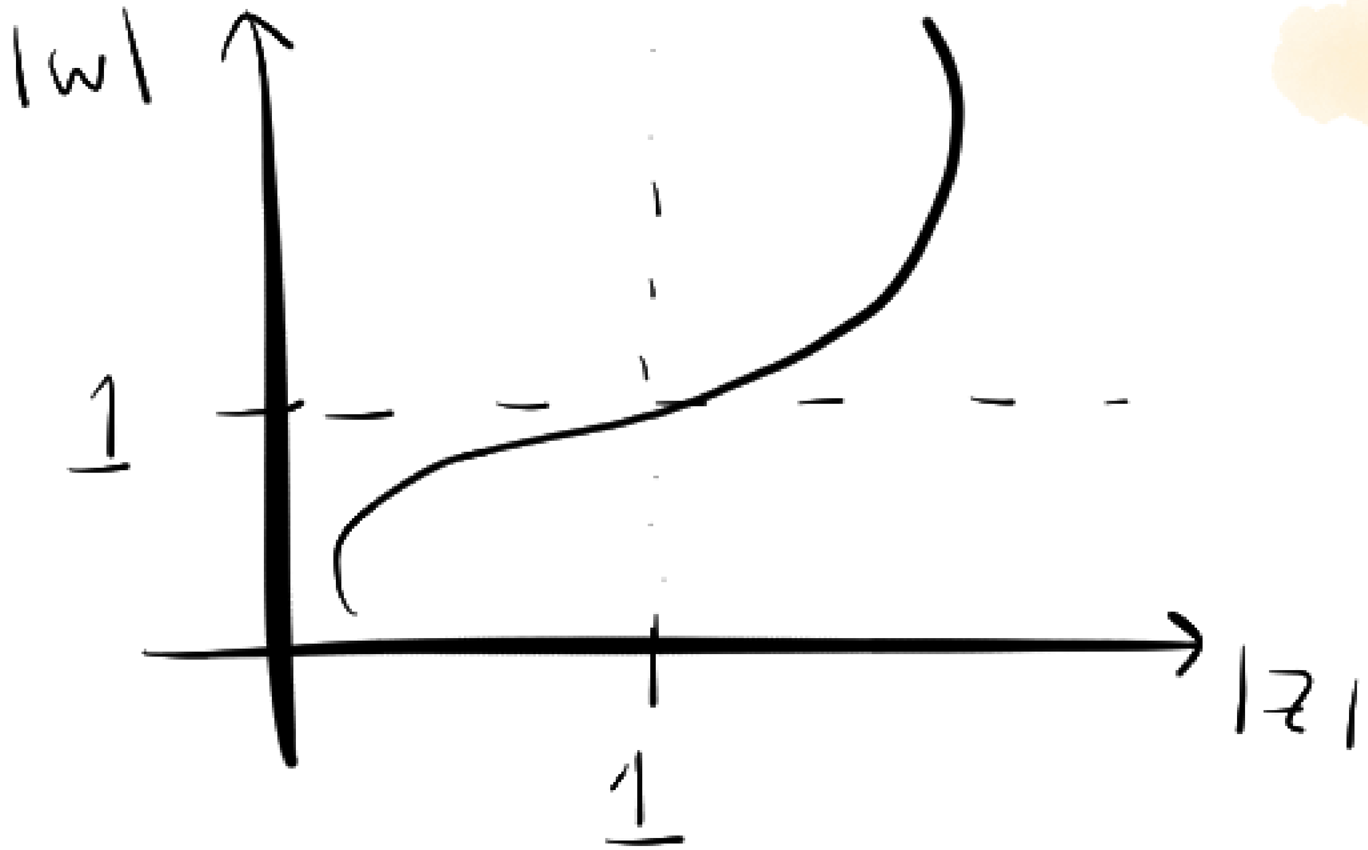
APPLICATION: "GENERALIZED DISTINGUISHED VARIETIES"

$p \in \mathbb{C}[z,w]$ $\quad$ $Z_p = \{(z,w) : p(z,w) = 0\}$

Z_p IS A DISTINGUISHED VARIETY

IF $Z_p \subset \mathbb{D}^2 \cup \overline{\mathbb{T}}^2 \cup \mathbb{E}^2$

$\mathbb{E} = \mathbb{C} \setminus \overline{\mathbb{D}}$



THM (AGLER–McCARTHY)

$\exists$ UNITARY MATRIX U

SUCH THAT

$$Z_p = \left\{ (z,w) : \det\left(U \begin{pmatrix} wI_m & 0 \\ 0 & I_n \end{pmatrix} - \begin{pmatrix} I_m & 0 \\ 0 & zI_n \end{pmatrix} \right) = 0 \right\}$$

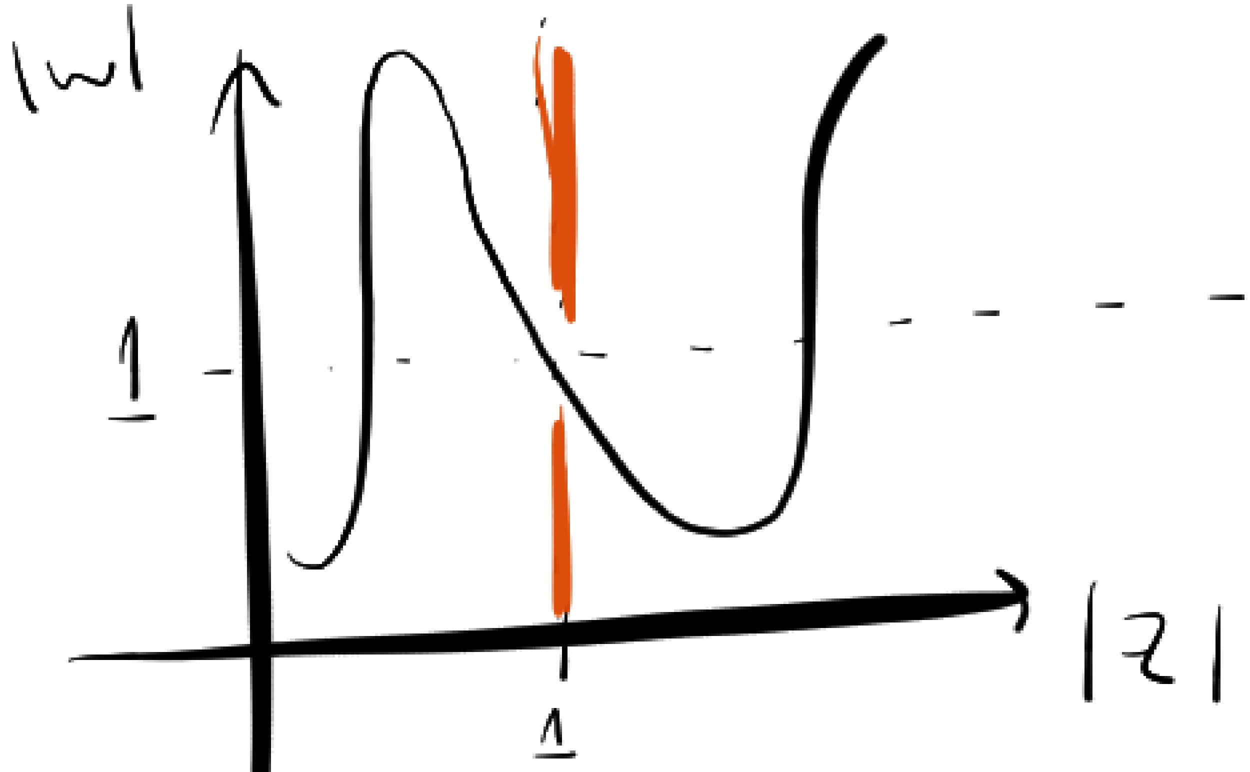
Z_p IS A GENERALIZED DISTINGUISHED VARIETY

IF

$$Z_p \subset (\mathbb{D} \times \mathbb{C}) \cup \mathbb{T}^2 \cup (\mathbb{E} \times \mathbb{C})$$

OR

$$Z_p \subset (\mathbb{C} \times \mathbb{D}) \cup \mathbb{T}^2 \cup (\mathbb{C} \times \mathbb{E})$$



THM (GIK) $\exists$ UNITARY MATRIX U

SUCH THAT

$$p(z,w) = \text{const}\,\det\left(U\begin{pmatrix} wI_m & & \\ & zI_{n_1} & \\ & & I_{n_2}\end{pmatrix} - \begin{pmatrix} I_m & & \\ & I_{n_1} & \\ & & zI_{n_2}\end{pmatrix}\right)$$

$n_2 = \#$ zeros of $p(z,0)$ in $\mathbb{D}$

$n_1 = n - n_2$

MORALS:

- POLYNOMIALS WITH NO ZEROS ON A FACE OF THE BIDISK ARE MUCH LIKE POLYS W/ NO ZEROS ON $\mathbb{D}^2$
- CHALLENGE TO UNDERSTAND POLYS W/ NO ZEROS ON $\overline{\mathbb{D}}^2$

FINE

APPENDIX

SKETCH OF PROOF 2:

$p \in \mathbb{C}[z,w]$ NO ZEROS ON $\overline{\mathbb{U}} \times \mathbb{D}$

$\gcd(p, \overleftarrow{p}) = 1$

$z \in \overline{\mathbb{U}}$

$$\frac{p(z,w)\overline{p(z,\eta)} - \overleftarrow{p}(z,w)\overline{\overleftarrow{p}(z,\eta)}}{1 - w\bar{\eta}}$$

$$= (1, \bar{\eta}, \ldots, \bar{\eta}^{m-1}) A(z) \begin{pmatrix} 1 \\ w \\ \vdots \\ w^{m-1} \end{pmatrix}$$

By Schur-Cohn

$A(z) > 0$ for all except finitely many $z \in \overline{\mathbb{U}}$.

1 VAR MATRIX FEJÉR-RIESZ THM

$$\Rightarrow A(z) = E(z)^* E(z), \quad E \in \mathbb{C}^{m \times m}[z]$$

$$\det E(z) \neq 0 \text{ FOR } z \in \mathbb{D}.$$

LET

$$E(z,w) = E(z) \begin{pmatrix} 1 \\ w \\ \vdots \\ w^m \end{pmatrix}$$

$$p(z,w)\overline{p(z,\eta)} - \tilde{p}(z,w)\overline{\tilde{p}(z,\eta)}$$

$$= (1 - w\bar{\eta}) E(z,\eta)^* E(z,w)$$

LURKING ISOMETRY!

For fixed $z \in \mathbb{T}$,

$$\begin{pmatrix} p(z,w) \\ wE(z,w) \end{pmatrix} \longmapsto \begin{pmatrix} \overleftarrow{p}(z,w) \\ E(z,w) \end{pmatrix}$$

extends to a unitary $U(z)$ on $\mathbb{C}^{m+1}$

$$U(z) = \begin{pmatrix} \overleftarrow{p}_m(z) & \cdots & \overleftarrow{p}_1(z) & \overleftarrow{p}_0(z) \\ & E(z) & & 0 \end{pmatrix} \begin{pmatrix} p_0(z) & p_1(z) & \cdots & p_m(z) \\ 0 & & E(z) & \end{pmatrix}^{-1}$$

Matrix valued rational function

unitary valued on $\mathbb{T}$

$$\frac{I - U(\zeta)^* U(z)}{1 - \bar{\zeta} z} = F(\zeta)^* F(z) - G(\zeta)^* G(z)$$

MATRIX RATIONAL FUNCTIONS

$$\Rightarrow \begin{pmatrix} I \\ zF(z) \\ G(z) \end{pmatrix} \longmapsto \begin{pmatrix} U(z) \\ F(z) \\ zG(z) \end{pmatrix}$$

EXTENDS TO A UNITARY V

DIMENSIONS DEPEND ON SMITH NORMAL FORM OF U

NOTE $$U(z)\begin{pmatrix} p(z,w) \\ wE(z,w) \end{pmatrix} = \begin{pmatrix} \overleftarrow{p}(z,w) \\ E(z,w) \end{pmatrix}$$

$$V\begin{pmatrix} I \\ zF(z) \\ G(z) \end{pmatrix}\begin{pmatrix} p(z,w) \\ wE(z,w) \end{pmatrix} = \begin{pmatrix} U(z) \\ F(z) \\ zG(z) \end{pmatrix}\begin{pmatrix} p(z,w) \\ wE(z,w) \end{pmatrix}$$

$$\rightsquigarrow \quad V\begin{pmatrix} p(z,w) \\ wE(z,w) \\ z\tilde{F}(z,w) \\ \tilde{G}(z,w) \end{pmatrix} = \begin{pmatrix} \overleftarrow{p}(z,w) \\ E(z,w) \\ \tilde{F}(z,w) \\ z\tilde{G}(z,w) \end{pmatrix}$$

$$\Longrightarrow$$

$$|p(z,w)|^2 + |w|^2 |E(z,w)|^2 + |z|^2 |\tilde{F}(z,w)|^2 + |\tilde{G}(z,w)|^2$$

$$= |\tilde{p}(z,w)|^2 + |E(z,w)|^2 + |\tilde{F}(z,w)|^2 + |z|^2 |\tilde{G}(z,w)|^2$$

$\Longrightarrow$ SUM OF SQUARES FORMULA

TECHNICALITIES:

- MAKING SURE $\tilde{F}, \tilde{G}$ ARE POLYNOMIAL

- GETTING THE CORRECT DIMENSIONS ON $\tilde{F}, \tilde{G}$

$\left(\text{I.E. } \tilde{G} \in \mathbb{C}^{n_2}[z,w], \quad n_2 = \#\text{ zeros of } p(z,0) \text{ IN } \mathbb{D} \right)$