Similarity of Cowen-Douglas Operators to Backward Bergman Shifts

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Ultimate Goal: Similarity Characterization of Cowen-Douglas Class Operators

$\mathcal{H}$: a separable Hilbert space

$$T \in \mathcal{L}(\mathcal{H})$$

$\mathbb{C}$: the complex plane

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

$$\operatorname{ran} T = \{Tx : x \in \mathcal{H}\} \; \ker T = \{x \in \mathcal{H} : Tx = 0\}$$

Operators T and $\tilde{T}$ are said to be *similar* if for some bounded, invertible operator A, we have $AT = \tilde{T}A$.

Let Ω be a domain in $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be in the *Cowen-Douglas class* $B_1(\Omega)$ if for every $\lambda \in \Omega$,

1. ran $(T - \lambda)$ is closed.
2. $\dim \ker (T - \lambda) = 1$.
3. $\bigvee_{\lambda \in \Omega} \ker (T - \lambda) = \mathcal{H}$.

Theorem (M. J. Cowen & R. G. Douglas)

For an operator T in this class, a Hermitian holomorphic eigenvector bundle

$$E_T = \coprod_{\lambda \in \Omega} \mathcal{E}(\lambda) = \{(\lambda, v_\lambda) : \lambda \in \Omega, v_\lambda \in \mathcal{E}(\lambda)\}$$

over Ω exists with the metric inherited from $\mathcal{H}$ and the natural projection, $\pi(\lambda, v_\lambda) = \lambda$.

Let $\mathcal{H}$ be a reproducing kernel Hilbert space with kernel $k_\lambda, \lambda \in \mathbb{D}$. Let $f, g \in \mathcal{H}$ and $z \in \mathbb{D}$.

$$M_z f = zf(z), \langle M_z f, g \rangle = \langle f, M_z^* g \rangle$$

$$M_z^* k_{\bar{\lambda}} = \lambda k_{\bar{\lambda}}$$

$$M_z^* \in B_1(\mathbb{D}) \text{ so that } E_{M_z^*} \text{ exists.}$$

The curvature κ_T of E_T for $T \in B_1(\mathbb{D})$ can be easily calculated as

$$\kappa_T = -\partial\bar{\partial} \log \|k_\lambda\|^2.$$

Theorem (H. Kwon & S. Treil)

Let $T \in B_1(\mathbb{D})$.

"Similarity of T, $\|T\| \leq 1$, on $\mathcal{H}$, to M_z^* on $H^2(\mathbb{D})$"

is equivalent to

"$\Delta\phi(z) = \kappa_{M_z^*}(z) - \kappa_T(z)$ for a bounded function ϕ defined on $\mathbb{D}$",

Similarity to M_z^* on $\mathcal{M}_n$?

$$\mathcal{M}_n = \{f \in \text{Hol}\,(\mathbb{D}) : \|f\|_n^2 = \sum_{j=0}^{\infty} |\hat{f}(j)|^2 \frac{1}{\binom{n+j-1}{j}} < \infty\}$$

$\mathcal{M}_1 = H^2(\mathbb{D})$
$\mathcal{M}_n = \{f \in \text{Hol}\,(\mathbb{D}) : (n-1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{n-2} dA(z) < \infty\} = A^2_{n-2}(\mathbb{D})$ for $n \geq 2$.
$\mathcal{M}_n$ is a reproducing kernel Hilbert space with reproducing kernel $k_\lambda^n = (1-\bar{\lambda}z)^{-n}$, $\lambda \in \mathbb{D}$.

$T \in \mathcal{L}(\mathcal{H})$ is called an *n-hypercontraction* if for all $1 \leq k \leq n$,

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} (T^*)^j T^j \geq 0.$$

Theorem (J. Agler)

An $n-$hypercontration $T \in \mathcal{L}(\mathcal{H})$ with $\lim_k \|T^k h\| = 0$ for every $h \in \mathcal{H}$ is unitarily equivalent to M_z^* restricted to an M_z^*-invariant subspace of a vector-valued space $\mathcal{M}_n$.

Using this Theorem, we conclude that E_T has the tensor product representation

$$\text{Ker } (T - \lambda) = \bigvee \{k_{\bar{\lambda}}\} \otimes \mathcal{N}(\lambda),$$

for some subspace $\mathcal{N}(\lambda)$.

Theorem (R. G. Douglas, H. Kwon, S. Treil)

Let $T \in B_1(\mathbb{D})$.

Similarity of an n*−hypercontraction* T *to* M_z^* *on* $\mathcal{M}_n$

is equivalent to

$\Delta\phi(z) \geq \kappa_{M_z^*}(z) - \kappa_T(z)$ *for some bounded function* ϕ

defined on $\mathbb{D}$.

$\kappa_{M_z^*}(z) = -\frac{n}{(1-|z|^2)^2}$

Ingredients of Proof:

Tensor product structure of E_T.

By Nikolski's Lemma, the corona problem is equivalent to the existence of an analytic projection.

Define projections onto $\bigvee\{k_{\bar{\lambda}}\}$ and $\mathcal{N}(\lambda)$.

The inner-outer factorization of a bounded, analytic function.