Extremal holomorphic maps and the symmetrised bidisc

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A similar notion of extremality, but with n equal to 2, occurs in the theory of hyperbolic complex spaces introduced by S. Kobayashi in 1977. In this context one studies the geometry and function theory of a domain $\Omega \subset \mathbb{C}^d$ with the aid of 2-extremal holomorphic maps from $\mathbb{D}$ to Ω.

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A prominent theme in hyperbolic complex geometry is a kind of duality between $\mathrm{Hol}(\mathbb{D}, \Omega)$ and $\mathrm{Hol}(\Omega, \mathbb{D})$, typified by the celebrated theorem of L. Lempert 1986, which in our terminology asserts that if Ω is convex then every 2-extremal map belonging to $\mathrm{Hol}(\mathbb{D}, \Omega)$ is a complex geodesic of Ω (that is, has an analytic left inverse).

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The notion of n-extremal map makes sense, however, in much greater generality.

n-extremal holomorphic maps

Definition 1. *Let* Ω *be a domain, let* $E \subset \mathbb{C}^N$*, let* $n \geq 1$*, let* $\lambda_1, \ldots, \lambda_n$ *be distinct points in* Ω *and let* $z_1, \ldots, z_n \in E$*. We say that the interpolation data*

$$\lambda_j \mapsto z_j : \Omega \to E, \quad j = 1, \ldots, n,$$

are extremally solvable if there exists a map $h \in \mathrm{Hol}(\Omega, E)$ *such that* $h(\lambda_j) = z_j$ *for* $j = 1, \ldots, n$*, but, for any open neighbourhood* U *of the closure of* Ω*, there is no* $f \in \mathrm{Hol}(U, E)$ *such that* $f(\lambda_j) = z_j$ *for* $j = 1, \ldots, n$*.*

Here $\mathrm{Hol}(\Omega, E)$ is the space of holomorphic maps from a domain Ω to a subset E.

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Here $\mathrm{Hol}(\Omega, E)$ is the space of holomorphic maps from a domain Ω to a subset E.

We say further that $h \in \mathrm{Hol}(\Omega, E)$ is n-extremal (for $\mathrm{Hol}(\Omega, E)$) if, for all choices of n distinct points $\lambda_1, \ldots, \lambda_n$ in Ω, the interpolation data

$$\lambda_j \mapsto h(\lambda_j) : \Omega \to E, \quad j = 1, \ldots, n,$$

are extremally solvable.

There are no 1-extremal holomorphic maps, so we shall always suppose that $n \geq 2$.

In this talk we shall be mainly concerned with n-extremals for $\mathrm{Hol}(\mathbb{D}, \Gamma)$ where the *symmetrised bidisc* $\mathbb{G}$ in $\mathbb{C}^2$ is defined to be the set

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and Γ is the closure of $\mathbb{G}$.

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Note that $\mathbb{G}$ is not isomorphic to any convex domain (Costara).

Interpolation in $\mathrm{Hol}(\mathbb{D}, \Gamma)$

The (finite) interpolation problem for $\mathrm{Hol}(\mathbb{D}, \Gamma)$ is the following:

Given Γ*-interpolation data*

$$\lambda_j \mapsto z_j, \qquad 1 \le j \le n, \tag{1}$$

where $\lambda_1, \ldots, \lambda_n$ *are* n *distinct points in the open unit disc* $\mathbb{D}$ *and* $z_1, \ldots, z_n$ *are* n *points in* Γ*, find if possible an analytic function*

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There is a satisfactory analytic theory of the problem (2) in the case that the number of interpolation points n is 2, but we are still far from understanding the problem for a general $n \in \mathbb{N}$.

Condition $\mathcal{C}_\nu$

Here we introduce a sequence of necessary conditions for the solvability of an n-point Γ-interpolation problem and put forward a conjecture about sufficiency. We will show here that these conditions are of strictly increasing strength.

Definition 2. *Corresponding to Γ-interpolation data*

$$\lambda_j \in \mathbb{D} \mapsto z_j = (s_j, p_j) \in \mathbb{G}, \qquad 1 \leq j \leq n, \tag{3}$$

we introduce:

Condition $\mathcal{C}_\nu(\lambda, z)$

For every Blaschke product υ of degree at most ν, the Nevanlinna-Pick data

$$\lambda_j \mapsto \Phi(\upsilon(\lambda_j), z_j) = \frac{2\upsilon(\lambda_j)p_j - s_j}{2 - \upsilon(\lambda_j)s_j}, \quad j = 1, \ldots, n, \tag{4}$$

are solvable.

Definition 3. *The function* Φ *is defined for* $(z, s, p) \in \mathbb{C}^3$ *such that* $zs \neq 2$ *by*

$$\Phi(z, s, p) = \frac{2zp - s}{2 - zs}.$$

We shall write $\Phi_z(s, p)$ *as a synonym for* $\Phi(z, s, p)$.

The Γ-interpolation conjecture

Conjecture 1. *Condition $\mathcal{C}_{n-2}$ is necessary and sufficient for the solvability of an n-point Γ-interpolation problem.*

Conjecture 1 is true in the case $n = 2$. We have no evidence for $n \geq 3$ and we are open minded as to whether or not it is likely to be true for all n.

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Observe that Pick's Theorem gives us an easily-checked criterion for the solvability of a Nevanlinna-Pick problem.

Proposition 1. *If $\lambda_j \mapsto z_j = (s_j, p_j)$, $\quad 1 \leq j \leq n$, are interpolation data for Γ then condition $\mathcal{C}_\nu(\lambda_1, \ldots, \lambda_n, z_1, \ldots, z_n)$ holds if and only if, for every Blaschke product υ of degree at most ν,*

$$\left[\frac{1 - \upsilon(\lambda_i) p_i \bar{p}_j \overline{\upsilon}(\lambda_j) - \frac{1}{2}\upsilon(\lambda_i)(s_i - p_i \bar{s}_j) - \frac{1}{2}(\bar{s}_j - \bar{p}_j s_i)\overline{\upsilon}(\lambda_j) - \frac{1}{4}(1 - \upsilon(\lambda_i)\bar{\upsilon}(\lambda_j)) s_i \bar{s}_j}{1 - \lambda_i \overline{\lambda}_j} \right]_{i,j=1}^{n} \tag{5}$$

is positive.

$\mathcal{C}_\nu$ are necessary

The conditions $\mathcal{C}_\nu$ are all necessary for the solvability of a Γ-interpolation problem.

Theorem 1. *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $z_j \in \mathbb{G}$ for $j = 1, 2, \ldots, n$.*

If there exists an analytic function

$$h : \mathbb{D} \to \Gamma$$

such that $h(\lambda_j) = z_j$ for $j = 1, 2, \ldots, n$ then, for any function υ in the Schur class $\mathcal{S} = \mathrm{Hol}(\mathbb{D}, \Delta)$, the Nevanlinna-Pick data

$$\lambda_j \mapsto \Phi(\upsilon(\lambda_j), z_j), \quad j = 1, \ldots, n, \tag{6}$$

are solvable. In particular, the condition $\mathcal{C}_\nu(\lambda, z)$ holds for every non-negative integer ν.

Extremality in Condition $\mathcal{C}_\nu$

To prove that condition $\mathcal{C}_\nu$ suffices for the solvability of an n-point Nevanlinna-Pick problem for Γ it is enough to prove it in the case that $\mathcal{C}_\nu$ holds *extremally*. Let us make this notion precise.

Recall that Γ-interpolation data $\lambda_j \mapsto z_j,\ 1 \le j \le n,$ are defined to satisfy condition $\mathcal{C}_\nu$ if, for every Blaschke product $\upsilon \in \mathcal{B}l_\nu$ of degree at most ν, the data

$$\lambda_j \mapsto \Phi(\upsilon(\lambda_j), z_j), \quad 1 \le j \le n, \tag{7}$$

are solvable for the classical Nevanlinna-Pick problem. If, in addition, there exists $m \in \mathcal{B}l_\nu$ such that the data

$$\lambda_j \mapsto \Phi(m(\lambda_j), z_j), \quad 1 \le j \le n,$$

are *extremally* solvable Nevanlinna-Pick data, then we shall say that the data $\lambda_j \mapsto z_j,\ 1 \le j \le n,$ *satisfy* $\mathcal{C}_\nu$ *extremally*, or the condition $\mathcal{C}_\nu(\lambda, z)$ *holds extremally*.

It is well known that Pick's criterion for the solvability of a classical Nevanlinna-Pick problem is expressible by an operator norm inequality; hence condition $\mathcal{C}_\nu$ can be expressed this way. Let

$$\mathcal{M} = \operatorname{span}\{K_{\lambda_1}, \ldots, K_{\lambda_n}\} \subset H^2, \tag{8}$$

where K is the Szegő kernel. Consider Γ-interpolation data

$$\lambda_j \mapsto z_j,\ 1 \le j \le n,$$

and introduce, for any function υ in the Schur class, the operator $X(\upsilon)$ on $\mathcal{M}$ given by

$$X(\upsilon)K_{\lambda_j} = \overline{\Phi(\upsilon(\lambda_j), z_j)}K_{\lambda_j}, \qquad 1 \le j \le n. \tag{9}$$

Pick's Theorem, as reformulated by Sarason, asserts that the Nevanlinna-Pick data

$$\lambda_j \mapsto \Phi(\upsilon(\lambda_j), z_j), \qquad 1 \le j \le n, \tag{10}$$

are solvable if and only if the operator $X(\upsilon)$ is a contraction. Furthermore, the Nevanlinna-Pick data (10) are extremally solvable if and only if $\|X(\upsilon)\| = 1$.

Thus $\mathcal{C}_\nu(\lambda, z)$ holds if and only if

$$\sup_{\upsilon \in \mathcal{B}l_\nu} \|X(\upsilon)\| \leq 1. \tag{11}$$

Proposition 2. *For any Γ-interpolation data $\lambda_j \mapsto z_j,\ 1 \leq j \leq n$, and $\nu \geq 0$, the following conditions are equivalent.*

(i) $\mathcal{C}_\nu(\lambda, z)$ *holds extremally;*

(ii) $\sup_{\upsilon \in \mathcal{B}l_\nu} \|X(\upsilon)\| = 1$;

(iii) $\mathcal{C}_\nu(\lambda, z)$ *holds and there exist $m \in \mathcal{B}l_\nu$ and $q \in \mathcal{B}l_{n-1}$ such that*

$$\Phi(m(\lambda_j), z_j) = q(\lambda_j), \quad j = 1, \ldots, n, \tag{12}$$

Moreover, when condition (iii) *is satisfied for some $m \in \mathcal{B}l_\nu$, there is a $unique$ $q \in \mathcal{B}l_{n-1}$ such that equations* (12) *hold. If, furthermore, the Γ-interpolation data $\lambda_j \mapsto z_j,\ 1 \leq j \leq n$, are solvable by an analytic function $h = (s, p) : \mathbb{D} \to \Gamma$, then*

$$\frac{2mp - s}{2 - ms} = q. \tag{13}$$

An auxiliary extremal for the condition $\mathcal{C}_\nu(\lambda, z)$

We shall say that any Blaschke product m with the properties described in Proposition 2(iii) is an *auxiliary extremal* for the condition $\mathcal{C}_\nu(\lambda, z)$.

Examples 2. Let $\lambda_1, \lambda_2, \lambda_3$ be any three distinct points in $\mathbb{D}$ and let $0 < r < 1$. In each of the following examples h is an analytic function from $\mathbb{D}$ to $\mathbb{G}$ and the data $\lambda_j \mapsto h(\lambda_j), 1 \leq j \leq 3,$ satisfy $\mathcal{C}_1$ extremally.

(1) Let $h(\lambda) = (2r\lambda, \lambda^2)$. Every degree 0 inner function $m \in \mathbb{T}$ is an auxiliary extremal for $\mathcal{C}_1$; there is no auxiliary extremal of degree 1.

(2) Let $h(\lambda) = (r(1 + \lambda), \lambda)$. Every $m \in \mathcal{B}l_1$ is an auxiliary extremal for $\mathcal{C}_1$. The corresponding q has degree $d(m) + 1$.

An auxiliary extremal for the condition $\mathcal{C}_\nu(\lambda, z)$

(3) Let

$$h(\lambda) = \left(2(1-r)\frac{\lambda^2}{1+r\lambda^3}, \frac{\lambda(\lambda^3+r)}{1+r\lambda^3}\right), \qquad \lambda \in \mathbb{D}.$$

The function $m(\lambda) = -\lambda$ is an auxiliary extremal for $\mathcal{C}_1$; there is no auxiliary extremal of degree 0. Here $q(\lambda) = -\lambda^2$.

(4) Let f be a Blaschke product of degree 1 or 2 and let $h = (2f, f^2)$. Every $m \in \mathcal{B}l_1$ is an auxiliary extremal and, for every m, we have $q = -f$.

Γ-inner functions

Definition 4. *A Γ-inner function is an analytic function $h : \mathbb{D} \to \Gamma$ such that the radial limit*

$$\lim_{r \to 1-} h(r\lambda) \in b\Gamma \tag{14}$$

for almost all $\lambda \in \mathbb{T}$.

Here $b\Gamma$ is the distinguished boundary of $\mathbb{G}$ (or Γ). It is the symmetrisation of the 2-torus:

$$b\Gamma = \{(z + w, zw) : |z| = |w| = 1\}.$$

By Fatou's Theorem, the radial limit (14) exists for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.

Observe that, if $h = (h_1, h_2)$ is a Γ-inner function, then h_2 is an inner function on $\mathbb{D}$ in the conventional sense.

The classes $\mathcal{E}_{\nu k}$

Proposition 2 tells us that if $h \in \text{Hol}(\mathbb{D}, \Gamma)$ and $\lambda_1, \dots, \lambda_n$ are distinct points in $\mathbb{D}$, then the Γ-interpolation data $\lambda_j \mapsto h(\lambda_j)$ satisfy $C_\nu(\lambda, h(\lambda))$ *extremally* if and only if there exists $m \in \mathcal{B}l_\nu$ such that $\Phi \circ (m, h) \in \mathcal{B}l_{n-1}$. This leads us to introduce the following classes of rational Γ-inner functions.

Definition 5. *For $\nu \geq 0$, $k \geq 1$ we say that the function h is in $\mathcal{E}_{\nu k}$ if $h = (s, p) \in \text{Hol}(\mathbb{D}, \Gamma)$ is rational and there exists $m \in \mathcal{B}l_\nu$ such that*

$$\frac{2mp - s}{2 - ms} \in \mathcal{B}l_{k-1}.$$

Remark 3. It is obvious that, for every $\nu \geq 0$,

$$\mathcal{E}_{\nu 1} \subset \mathcal{E}_{\nu 2} \subset \cdots \subset \mathcal{E}_{\nu k} \subset \mathcal{E}_{\nu, k+1} \subset \dots,$$

and, for every $k \geq 1$,

$$\mathcal{E}_{0k} \subset \mathcal{E}_{1k} \subset \cdots \subset \mathcal{E}_{\nu k} \subset \mathcal{E}_{\nu+1, k} \subset \dots.$$

Superficial Γ-inner functions and the classes $\mathcal{E}_{\nu 1}$

For any inner function φ and $\omega \in \mathbb{T}$ the function $h = (\omega + \varphi, \omega\varphi)$ is Γ-inner, and has the property that $h(\lambda)$ lies in the topological boundary $\partial\Gamma$ of Γ for all $\lambda \in \mathbb{D}$.

Recall that $(s,p) \in \partial\Gamma \quad \Leftrightarrow |s| \leq 2$ and $|s - \bar{s}p| = 1 - |p|^2$

$\Leftrightarrow$ there exist $z \in \mathbb{T}$ and $w \in \Delta$ such that $s = z + w, \ p = zw.$

Definition 6. *A function $h \in \mathrm{Hol}(\mathbb{D}, \Gamma)$ is superficial if $h(\mathbb{D}) \subset \partial\Gamma$.*

The image of a function in $\mathrm{Hol}(\mathbb{D}, \Gamma)$ is either contained in or disjoint from $\partial\Gamma$.

Lemma 1. *If $h \in \mathrm{Hol}(\mathbb{D}, \Gamma)$ is not superficial then $h(\mathbb{D}) \subset \mathbb{G}$.*

Proposition 3. *A Γ-inner function h is superficial if and only if there is an $\omega \in \mathbb{T}$ and an inner function p such that $h = (\omega p + \bar{\omega}, p)$.*

Theorem 4. *For every $\nu \geq 1$, the class $\mathcal{E}_{\nu 1}$ is equal to $\mathcal{E}_{01}$ and consists of the superficial rational Γ-inner functions.*

The classes $\mathcal{E}_{\nu k}$ and k-extremals, $k \geq 2$

Theorem 5. *If $h \in \mathcal{E}_{\nu k}$, where $\nu \geq 0$ and $k \geq 2$, and h is not superficial then h is k-extremal for $\mathrm{Hol}(\mathbb{D}, \Gamma)$.*

If Conjecture 1 is true then all n-extremals for Γ lie in $\mathcal{E}_{n-2,n}$.

Observation 6. *Let $n \geq 2$. If condition $\mathcal{C}_{n-2}$ suffices for the solvability of n-point Γ-interpolation problems then every rational Γ-inner function h which is n-extremal for $\mathrm{Hol}(\mathbb{D}, \Gamma)$ belongs to $\mathcal{E}_{n-2,n}$.*

Complex geodesics of $\mathbb{G}$ and the classes $\mathcal{E}_{\nu 2}$

We recall that an analytic function $h : \mathbb{D} \to \Omega$ is called a *complex geodesic of* Ω if there exists an analytic left inverse $g : \Omega \to \mathbb{D}$ of h.

Example 1. *Let $|\beta| < 1$. The function*

$$h(\lambda) = (\beta\lambda + \bar{\beta}, \lambda) \tag{15}$$

is not only Γ-inner – it is a complex geodesic of $\mathbb{G}$. The simplest left inverse is the projection $(s, p) \mapsto p$. The domain $\mathbb{G}$ also has complex geodesics of degree 2.

Proposition 4. *An analytic function $h : \mathbb{D} \to \mathbb{G}$ is a complex geodesic of $\mathbb{G}$ if and only if there is an $\omega \in \mathbb{T}$ such that $\Phi_\omega \circ h \in \mathrm{Aut}\ \mathbb{D}$. Furthermore, every complex geodesic of $\mathbb{G}$ is Γ-inner.*

Theorem 7. *For $\nu \geq 0$ the set $\mathcal{E}_{\nu 2}$ is the union of the set of superficial rational Γ-inner functions and the set of complex geodesics of $\mathbb{G}$.*

Condition $\mathcal{C}_\nu$ and the classes $\mathcal{E}_{\nu k}$

It is clear that $\mathcal{C}_\nu(\lambda, z)$ implies $\mathcal{C}_{\nu-1}(\lambda, z)$ for any Γ-interpolation data $\lambda \mapsto z$. To show that $\mathcal{C}_\nu$ is *strictly stronger* than $\mathcal{C}_{\nu-1}$ we need to find Γ-interpolation data

$$\lambda_j \in \mathbb{D} \mapsto z_j = (s_j, p_j) \in \mathbb{G}, \qquad 1 \le j \le k, \tag{16}$$

such that
(i) for every Blaschke product υ of degree at most $\nu - 1$,

$$\lambda_j \mapsto \frac{2\upsilon(\lambda_j)p_j - s_j}{2 - \upsilon(\lambda_j)s_j}, \quad j = 1, \dots, k, \tag{17}$$

are solvable Nevanlinna-Pick data, but
(ii) there is a Blaschke product m of degree ν such that

$$\lambda_j \mapsto \frac{2m(\lambda_j)p_j - s_j}{2 - m(\lambda_j)s_j}, \quad j = 1, \dots, k, \tag{18}$$

are not solvable Nevanlinna-Pick data.

Condition $\mathcal{C}_\nu$ and the classes $\mathcal{E}_{\nu k}$

For distinct points $\lambda_1, \dots, \lambda_k$ in $\mathbb{D}$, we define

$$\mathrm{Solv}(\lambda_1, \dots, \lambda_k) = \{(f(\lambda_1), \dots, f(\lambda_k)) \in \mathbb{D}^k : f \in \mathcal{S}\},$$

and

$$\mathrm{Unsolv}(\lambda_1, \dots, \lambda_k) = \mathbb{C}^k \setminus \mathrm{Solv}(\lambda_1, \dots, \lambda_k).$$

Thus $w = (w_1, \dots, w_k) \in \mathrm{Solv}(\lambda_1, \dots, \lambda_k)$ if and only if $\lambda_j \mapsto w_j,\ j = 1, \dots, k$, are solvable Nevanlinna-Pick data.

Proposition 5. *Let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$.*

(i) $\mathrm{Solv}(\lambda_1, \dots, \lambda_n)$ *is closed in $\mathbb{C}^n$.*

(ii) *Let $w = (w_1, \dots, w_n) \in \mathrm{Solv}(\lambda_1, \dots, \lambda_n)$. The Nevanlinna-Pick data $\lambda_j \mapsto w_j$, $j = 1, \dots, n$, are extremally solvable if and only if $w \in \partial\mathrm{Solv}(\lambda_1, \dots, \lambda_n)$.*

Proposition 6. *If there exists a nonconstant function $h \in \mathcal{E}_{\nu k} \setminus \mathcal{E}_{\nu-1,k}$ then $\mathcal{C}_\nu$ is strictly stronger than $\mathcal{C}_{\nu-1}$. In fact there is a set of Γ-interpolation data $\lambda_j \mapsto z_j$ with k interpolation points which satisfies $\mathcal{C}_{\nu-1}$ but not $\mathcal{C}_\nu$.*

Inequations for the classes $\mathcal{E}_{\nu k}$

In order to apply Proposition 6 we must establish the strict inclusion

$$\mathcal{E}_{\nu-1,k} \subsetneq \mathcal{E}_{\nu,k}$$

for a suitable k.

Proposition 7. *For all $\nu \geq 1$ and $0 < r < 1$, the function*

$$h_\nu(\lambda) = \left(2(1-r)\frac{\lambda^{\nu+1}}{1+r\lambda^{2\nu+1}}, \frac{\lambda(\lambda^{2\nu+1}+r)}{1+r\lambda^{2\nu+1}} \right), \ \lambda \in \mathbb{D}, \tag{19}$$

belongs to $\mathcal{E}_{\nu,\nu+2} \setminus \mathcal{E}_{\nu-1,\nu+2}$.

Proof. It is clear that h_ν is analytic on Δ. Let $h_\nu = (s, p)$. It is simple to check that $s = \bar{s}p$ on $\mathbb{T}$, that $|s| \leq 2$ on $\mathbb{T}$ and that $|p(\lambda)| = 1$ on $\mathbb{T}$. This implies that $h_\nu(\mathbb{T}) \subset b\Gamma$ and that h_ν is Γ-inner.

Let $m(\lambda) = -\lambda^\nu$, so that $m \in \mathcal{B}l_\nu$. It is simple to verify that

$$\Phi \circ (m, h_\nu) = \frac{2mp - s}{2 - ms}(\lambda) = -\lambda^{\nu+1} \in \mathcal{B}l_{\nu+1},$$

and so $h_\nu \in \mathcal{E}_{\nu,\nu+2}$.

To prove that h_ν is not in $\mathcal{E}_{\nu-1,\nu+2}$ we must show that, for all $\upsilon \in \mathcal{B}l_{\nu-1}$, the Blaschke product $\Phi \circ (\upsilon, h_\nu)$ has degree at least $\nu + 2$. We can do it using cancellations in the functions $\Phi \circ (\upsilon, h_\nu)$. It transpires that cancellations can only happen at special points on the unit circle: $\lambda^{2\nu+1} = -1$.

$\mathcal{C}_\nu$ is strictly stronger than $\mathcal{C}_{\nu-1}$

Our main theorem follows easily.

Theorem 8. *For all $\nu \geq 1$, the condition $\mathcal{C}_\nu$ is strictly stronger than $\mathcal{C}_{\nu-1}$. In fact there is a set of Γ-interpolation data $\lambda_j \mapsto z_j$ with $\nu + 2$ interpolation points which satisfies $\mathcal{C}_{\nu-1}$ but not $\mathcal{C}_\nu$.*

As we observed above, $\mathcal{C}_0$ is necessary and sufficient for solvability of a Γ-interpolation problem when $n = 2$, but a consequence of Theorem 8 is:

Corollary 1. *For all $n \geq 3$, Condition $\mathcal{C}_{n-3}$ does not suffice for the solvability of an n-point Γ-interpolation problem.*

Table of relations between the classes $\mathcal{E}_{\nu k}$

$$
\begin{array}{ccccccccccccccc}
\mathcal{E}_{01} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{02} & \overset{(1)}{\subsetneq} & \mathcal{E}_{03} & \overset{(1)}{\subsetneq} & \mathcal{E}_{04} & \overset{(1)}{\subsetneq} & \mathcal{E}_{05} & \overset{(1)}{\subsetneq} & \mathcal{E}_{06} & \overset{(1)}{\subsetneq} & \mathcal{E}_{07} & \subsetneq & \ldots \\
\overset{(4)}{\|} & & \overset{(5)}{\|} & & \overset{(2)}{\not\cap} & & \cap & & \cap & & \overset{(3)}{\not\cap} & & \overset{(3)}{\not\cap} & & \\
\mathcal{E}_{11} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{12} & \overset{(1)}{\subsetneq} & \mathcal{E}_{13} & \overset{(1)}{\subsetneq} & \mathcal{E}_{14} & \overset{(1)}{\subsetneq} & \mathcal{E}_{15} & \overset{(1)}{\subsetneq} & \mathcal{E}_{16} & \overset{(1)}{\subsetneq} & \mathcal{E}_{17} & \subsetneq & \ldots \\
\overset{(4)}{\|} & & \overset{(5)}{\|} & & \cap & & \overset{(2)}{\not\cap} & & \cap & & \cap & & \cap & & \\
\mathcal{E}_{21} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{22} & \overset{(1)}{\subsetneq} & \mathcal{E}_{23} & \overset{(1)}{\subsetneq} & \mathcal{E}_{24} & \overset{(1)}{\subsetneq} & \mathcal{E}_{25} & \overset{(1)}{\subsetneq} & \mathcal{E}_{26} & \overset{(1)}{\subsetneq} & \mathcal{E}_{27} & \subsetneq & \ldots \\
\overset{(4)}{\|} & & \overset{(5)}{\|} & & \cap & & \cap & & \overset{(2)}{\not\cap} & & \cap & & \cap & & \\
\mathcal{E}_{31} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{32} & \subset & \mathcal{E}_{33} & \overset{(1)}{\subsetneq} & \mathcal{E}_{34} & \overset{(1)}{\subsetneq} & \mathcal{E}_{35} & \overset{(1)}{\subsetneq} & \mathcal{E}_{36} & \overset{(1)}{\subsetneq} & \mathcal{E}_{37} & \subsetneq & \ldots \\
\overset{(4)}{\|} & & \overset{(5)}{\|} & & \cap & & \cap & & \cap & & \overset{(2)}{\not\cap} & & \cap & & \\
\mathcal{E}_{41} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{42} & \subset & \mathcal{E}_{43} & \subset & \mathcal{E}_{44} & \overset{(1)}{\subsetneq} & \mathcal{E}_{45} & \overset{(1)}{\subsetneq} & \mathcal{E}_{46} & \overset{(1)}{\subsetneq} & \mathcal{E}_{47} & \subsetneq & \ldots \\
\overset{(4)}{\|} & & \overset{(5)}{\|} & & \cap & & \cap & & \cap & & \cap & & \overset{(2)}{\not\cap} & & \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & &
\end{array}
$$

References

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Thank you