# Hardy-type spaces and harmonic Bergman spaces on the hyperbolic disc

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# Introduction

Joint with G. Mauceri and M. Vallarino

## Setting

Class of Riemannian manifolds $\supset$ hyperbolic disc (all symmetric spaces of the noncompact type)

## Aim

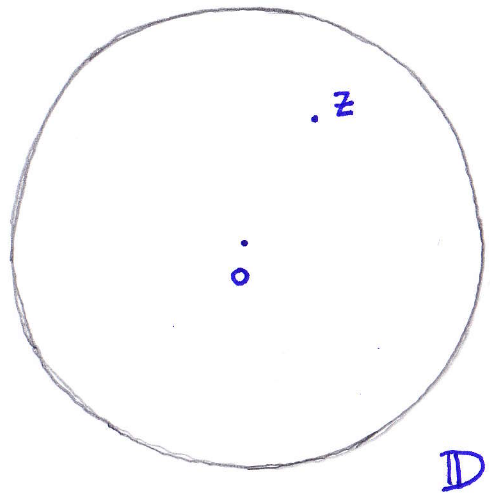
Find analogues of $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ on these manifolds

# THE HYPERBOLIC DISC

$$\mathbb{D} = SU(1,1) / S(U(1) \times U(1))$$

$$d(z,0) = \frac{1}{2} \log \frac{1+|z|}{1-|z|}$$

$$d\mu(z) = \frac{dx\, dy}{(1-|z|^2)^2}$$




$$\mathcal{L}_0 = -(1-|z|^2)^2 \Delta$$

essentially self-adjoint on $C_c^\infty(\mathbb{D}) \subset L^2(\mu)$

# MODEL OPERATORS

$\nabla \mathcal{L}^{-1/2}$ Riesz transform

$\mathcal{L}^{iu}$ ($u \neq 0$ real) purely imaginary powers of $\mathcal{L}$

$\left.\begin{array}{l} \text{Riesz transform} \\ \text{purely imaginary powers of } \mathcal{L} \end{array}\right\} \begin{array}{l} \text{bounded on } L^p(\mu), \quad 1<p<\infty \\ \text{unbounded on } L^1(\mu) \end{array}$

Spherical analysis on $SU(1,1) \Rightarrow \exists\, c>0$ s.t.

$$|k_{\nabla\mathcal{L}^{-1/2}}(z,0)| \asymp \begin{cases} d(z,0)^{-2} \\ d(z,0)^{-1/2}\, e^{-2d(z,0)} \end{cases} \qquad |\nabla k_{\nabla\mathcal{L}^{-1/2}}(z,0)| \asymp \begin{cases} d(z,0)^{-3} & z \text{ near } 0 \\ d(z,0)^{-1/2}\, e^{-2d(z,0)} & z \text{ near } \partial\mathbb{D} \end{cases}$$

$k_{\nabla\mathcal{L}^{-1/2}}$ behaves like a standard kernel near $0$

$k_{\nabla\mathcal{L}^{-1/2}}$ and $|\nabla k_{\nabla\mathcal{L}^{-1/2}}|$ are nonintegrable at infinity

# THE PROBLEM

**Find** $X^1 \subset L^1(\mu)$ s.t.

(i) the model operators are bounded from $X^1$ to $L^1(\mu)$

(ii) $X^1$ interpolates with $L^2(\mu)$ to give $L^p(\mu)$, $1<p<2$

(iii) the dual $Y^1$ of $X^1$ "behaves much like" BMO

**Caveat** $\mathbb{D}$ is not a space of homogeneous type in the sense of Coifman-Weiss

$$H^1(\mu) := \{ f = \sum c_j a_j : \ a_j \text{ CW-atom}, \ \sum |c_j| < \infty \}$$

$a$ is a CW-atom in $\mathbb{D}$ if

(1) $\operatorname{supp} a \subset B$

(2) $\|a\|_2 \leqslant \mu(B)^{-1/2}$

(3) $\int_B a \, d\mu = 0$

## $H^1(\mu)$ is inadequate for our purposes

**PROPOSITION** Set $a := \mathbb{1}_B - \mathbb{1}_{B'}$. Then $|\nabla \mathcal{L}^{-1/2} a| \notin L^1(\mu)$.

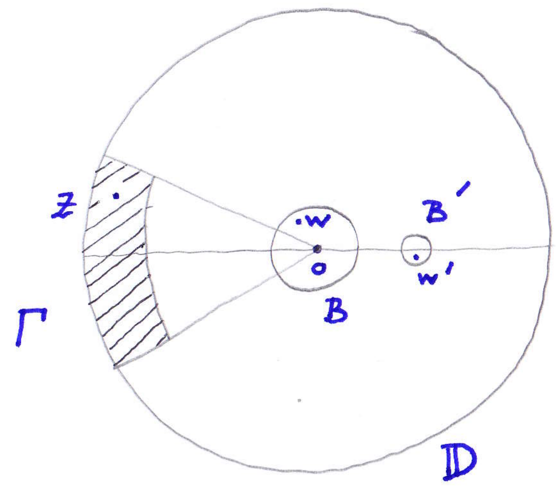
**PROOF** By the Federer–Fleming inequality

$$\|\nabla \mathcal{L}^{-1/2} a\|_1 \geq c \, \|\mathcal{L}^{-1/2} a\|_1$$

Observe that

$$\mathcal{L}^{-1/2} a(z) = \int_B K(z,w)\, d\mu(w) - \int_{B'} K(z,w')\, d\mu(w')$$

$$\geq C \, K(z,0) \quad \forall z \in \Gamma$$




$$\Rightarrow \|\mathcal{L}^{-1/2} a\|_1 \geq C \int_\Gamma K(z,0)\, d\mu(z) = +\infty$$

$\square$

# NEW SPACES

$$X^1 := \left\{ f = \sum c_j A_j : \quad A_j \text{ special atom}, \ \sum |c_j| < \infty \right\}$$

A is a special atom if

(1) $\operatorname{supp} A \subset B$ and $r_B \leq 1$

(2) $\|A\|_2 \leq \mu(B)^{-1/2}$

(3) $\int_B A \cdot H \, d\mu = 0 \quad \forall H \in b^2(B)$

where

$$b^2(B) := \left\{ H \in L^2(B) : \ \mathcal{L} H = 0 \right\}$$

is the harmonic Bergman space.

Note that

(3) $\Longleftrightarrow$ (3') $\int_B A \cdot H \, d\mu = 0$ for every global $\mathcal{L}$-harmonic function $H$

$\pi_B$ harmonic Bergman projector of $L^2(B)$ onto $b^2(B)$

$$Y^1 := \left\{ F \in L^2_{loc}(\mu) : \|F\|_{Y^1} := \sup_{B:\, r_B \leq 1} \left[ \frac{1}{\mu(B)} \int_B |F - \pi_B F|^2 \, d\mu \right]^{1/2} < \infty \right\}$$

THEOREM [Mauceri, M., Vallarino]

(i) The model operators are bounded from $X^1$ to $L^1(\mu)$

(ii) if $1/p = 1 - \theta/2$, then $[X^1, L^2(\mu)]_\theta = L^p(\mu)$

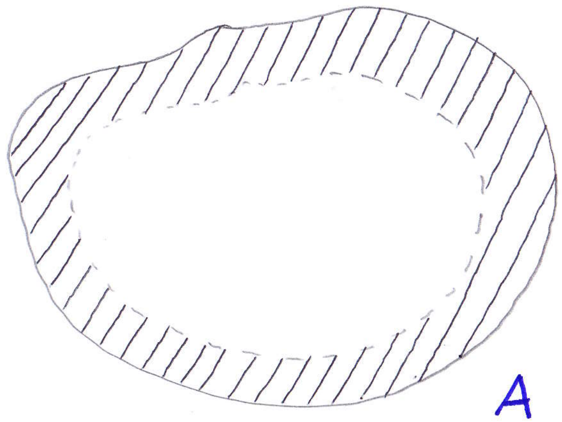
(iii) $Y^1$ is the dual space of $X^1$

# A CLASS OF MANIFOLDS

We assume:

(i) bounded geometry [ - positive injectivity radius
- Ricci curvature bounded from below ]

(ii) $\underbrace{h(M)}_{\text{cheeger's constant}} := \inf \frac{\sigma(\partial A)}{\mu(A)} > 0$



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**THEOREM** If M has bounded geometry, then

$h(M) > 0 \iff \underbrace{b}_{\text{Spectral gap}} := \min Sp_2(\Delta) > 0$

## Examples

(a) $M$ noncompact semisimple Lie group with finite centre, e.g. $SU(1,1)$, with any invariant metric [$M$ has spectral gap, because it is nonamenable]

(b) $M$ is a Damek-Ricci space or a symmetric space of the noncompact type with the Killing metric

(c) $M$ is a Cartan-Hadamard manifold with spectral gap

(d) $M = \{x+iy : y>0\}$, $0<a<b<2a$

$$ds^2 = \frac{1}{y^2} \, \frac{b^{-2} y + a^{-2}}{y+1} \, (dx^2+dy^2) \asymp \begin{cases} \dfrac{dx^2+dy^2}{(by)^2} & y \gg 1 \\[2ex] \dfrac{dx^2+dy^2}{(ay)^2} & y \ll 1 \end{cases}$$

Curiosity: the centred Hardy-Littlewood maximal function is bounded on $L^p(M)$ if $p > \frac{b}{a}$ and it is unbounded if $p < \frac{b}{a}$.

THANK YOU

FOR

YOUR ATTENTION!