On the reproducing kernel thesis for operators in Bergman-type spaces

Mishko Mitkovski
joint work with
Brett Wick

Department of Mathematical Sciences
Clemson University

May 24, 2013

Theorem (Suarez, 2007)

Let T be a bounded operator on the Bergman space of the ball. If T belongs in the Toeplitz algebra and its Berezin transform vanishes at the boundary then T must be compact.

Theorem (Suarez, 2007)

Let T be a bounded operator on the Bergman space of the ball. If T belongs in the Toeplitz algebra and its Berezin transform vanishes at the boundary then T must be compact.

Axler and Zheng proved it for operators in the algebraic part of the algebra.

Theorem (Suarez, 2007)

Let T be a bounded operator on the Bergman space of the ball. If T belongs in the Toeplitz algebra and its Berezin transform vanishes at the boundary then T must be compact.

Axler and Zheng proved it for operators in the algebraic part of the algebra.

Englis for Bergman spaces on bounded symmetric domains.

Theorem (Suarez, 2007)

Let T be a bounded operator on the Bergman space of the ball. If T belongs in the Toeplitz algebra and its Berezin transform vanishes at the boundary then T must be compact.

Axler and Zheng proved it for operators in the algebraic part of the algebra.

Englis for Bergman spaces on bounded symmetric domains.

Bauer and Isralowitz for Bargmann-Fock space.

Reproducing Kernel Thesis

Let $\mathcal{B}(\Omega)$ be a reproducing kernel Hilbert space (RKHS)

- For bounded operators:

  If $\sup_z \|Tk_z\| < \infty$ and $\sup_z \|T^* k_z\| < \infty$, then T is bounded.

  Stronger version:
  If $\langle Tk_z, k_z \rangle$ is bounded then T is bounded.

Reproducing Kernel Thesis

Let $\mathcal{B}(\Omega)$ be a reproducing kernel Hilbert space (RKHS)

- For bounded operators:

  If $\sup_z \|Tk_z\| < \infty$ and $\sup_z \|T^*k_z\| < \infty$, then T is bounded.

  Stronger version:
  If $\langle Tk_z, k_z\rangle$ is bounded then T is bounded.

- For compact operators:

  $k_z \to 0$ weakly $\implies$ $\|Tk_z\| \to 0$ then T is compact.

  Weaker version:
  $k_z \to 0$ weakly $\implies$ $\langle Tk_z, k_z\rangle \to 0$ then T is compact.

  Berezin transform: $\tilde{T}(z) = \langle Tk_z, k_z\rangle$.

General (but weaker) result

(Nordgren, Rosenthal 94') Let $\mathcal{K}(\Omega)$ be a RKHS such that

$k_z \to 0$ weakly whenever $z \to \xi \in \partial\Omega$.

Then T is compact if and only if for every unitary U

$$\langle TUk_z, Uk_z \rangle \to 0,$$

whenever $z \to \xi \in \partial\Omega$.

Reproducing kernel Hilbert spaces

We assume that $\mathcal{B}(\Omega)$ has the following properties:
Property 1: Ω domain in $\mathbb{C}^N$ possessing the following type of symmetries $\phi_z \in Aut(\Omega)$, $z \in \Omega$

- $\phi_z(0) = z$
- $\phi_z(\phi_z(w)) = w$

Reproducing kernel Hilbert spaces

We assume that $\mathcal{B}(\Omega)$ has the following properties:
Property 1: Ω domain in $\mathbb{C}^N$ possessing the following type of symmetries $\phi_z \in Aut(\Omega)$, $z \in \Omega$

- $\phi_z(0) = z$
- $\phi_z(\phi_z(w)) = w$

Example:
If $\Omega = \mathbb{C}$ then $\phi_z(w) = z - w$

If $\Omega = \mathbb{D}$ then $\phi_z(w) = \frac{z-w}{1-\bar{z}w}$

Reproducing kernel Hilbert spaces

Property 2: There is a metric d on Ω which is invariant under all ϕ_z.

Reproducing kernel Hilbert spaces

Property 2: There is a metric d on Ω which is invariant under all ϕ_z.

Property 3: There is a measure $d\lambda$ on Ω which is invariant under all ϕ_z.

Reproducing kernel Hilbert spaces

Property 2: There is a metric d on Ω which is invariant under all ϕ_z.

Property 3: There is a measure $d\lambda$ on Ω which is invariant under all ϕ_z.

Property 4:

$$f = \int_\Omega \langle f, k_z \rangle \, k_z d\lambda(z).$$

$$\|f\|^2 = \int_\Omega |\langle f, k_z \rangle|^2 d\lambda(z).$$

Loosely speaking: $\{k_z\}_{z\in\Omega}$ forms a continuously indexed o.n.b. for $\mathcal{B}(\Omega)$.

Property 5: $|\langle k_z, k_w\rangle| = o(1)$ as $d(z,w) \to \infty$

Examples

Main examples:

1. Bergman Space:
$\Omega = \mathbb{B}_n$; $d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$; The metric d is the Bergman metric.

2. Bargmann-Fock space:
$\Omega = \mathbb{C}^n$; $d\lambda(z)$ Lebesgue area measure; The metric d is the Euclidian metric.

Are the following true?

If $\sup_{z\in\Omega}\|Tk_z\| < \infty$ and $\sup_{z\in\Omega}\|T^*k_z\| < \infty$ then is T bounded?

If $\|Tk_z\| \to 0$, then T is compact; If $\langle Tk_z, k_z\rangle \to 0$, then is T compact?

Are the following true?

If $\sup_{z\in\Omega} \|Tk_z\| < \infty$ and $\sup_{z\in\Omega} \|T^* k_z\| < \infty$ then is T bounded?

If $\|Tk_z\| \to 0$, then T is compact; If $\langle Tk_z, k_z\rangle \to 0$, then is T compact?

Theorem

Let $T : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ be a linear operator defined a priori only on the linear span of the normalized reproducing kernels of $\mathcal{B}(\Omega)$. Define T^* on the same set by duality. If

$$\sup_{z\in\Omega} \|U_z T k_z\|_{L^p(\Omega;d\sigma)} < \infty \quad \text{and} \quad \sup_{z\in\Omega} \|U_z T^* k_z\|_{L^p(\Omega;d\sigma)} < \infty$$

for some $p > \frac{4-\kappa}{2-\kappa}$ then T is bounded on $\mathcal{B}(\Omega)$.

where $d\sigma(z) = d\lambda(z)/\|K_z\|^2$ and $U_z f(w) = f(\phi_z(w))k_z(w)$.

Are the following true?

If $\sup_{z\in\Omega} \|Tk_z\| < \infty$ and $\sup_{z\in\Omega} \|T^* k_z\| < \infty$ then is T bounded?

If $\|Tk_z\| \to 0$, then T is compact; If $\langle Tk_z, k_z\rangle \to 0$, then is T compact?

Theorem

Let $T : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ be a linear operator defined a priori only on the linear span of the normalized reproducing kernels of $\mathcal{B}(\Omega)$. Define T^* on the same set by duality. If

$$\sup_{z\in\Omega} \|U_z T k_z\|_{L^p(\Omega;d\sigma)} < \infty \quad \text{and} \quad \sup_{z\in\Omega} \|U_z T^* k_z\|_{L^p(\Omega;d\sigma)} < \infty$$

for some $p > \frac{4-\kappa}{2-\kappa}$ then T is bounded on $\mathcal{B}(\Omega)$.

where $d\sigma(z) = d\lambda(z)/\|K_z\|^2$ and $U_z f(w) = f(\phi_z(w))k_z(w)$.
Cao, Wang, Zhu 2012 for the classical Bargmann-Fock space.

Theorem

Let $T : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ *be a linear operator defined a priori only on the linear span of the normalized reproducing kernels of* $\mathcal{B}(\Omega)$. *Define* T^* *on the same set by duality. If*

$$\int_\Omega |\langle Tk_z, k_w\rangle|\, d\lambda(w) < \infty, \quad \int_\Omega |\langle T^* k_z, k_w\rangle|\, d\lambda(w) < \infty$$

then T *is bounded on* $\mathcal{B}(\Omega)$.

Theorem

Let $T : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ be a linear operator. If

$$\sup_{z\in\Omega} \|U_z T k_z\|_{L^p(\Omega;d\sigma)} < \infty \quad \text{and} \quad \sup_{z\in\Omega} \|U_z T^* k_z\|_{L^p(\Omega;d\sigma)} < \infty,$$

for some $p > \frac{4-\kappa}{2-\kappa}$, then

(a) $\|T\|_e \simeq \sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|U_z^* T U_z f\|$.

(b) *If $\lim_{d(z,0)\to\infty} \|Tk_z\| = 0$ then T must be compact.*

Every Toeplitz operator T satisfies the conditions above.

Theorem

Let $T : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ be a linear operator. If

$$\sup_{z\in\Omega} \|U_z T k_z\|_{L^p(\Omega;d\sigma)} < \infty \quad \text{and} \quad \sup_{z\in\Omega} \|U_z T^* k_z\|_{L^p(\Omega;d\sigma)} < \infty,$$

for some $p > \frac{4-\kappa}{2-\kappa}$, then

(a) $\|T\|_e \simeq \sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|U_z^* T U_z f\|$.

(b) *If* $\lim_{d(z,0)\to\infty} \|Tk_z\| = 0$ *then T must be compact.*

Every Toeplitz operator T satisfies the conditions above.

Suarez 2007, $\mathcal{B}(\Omega) =$ classical Bergman space and T in the Toeplitz algebra.

Bauer and Isralowitz 2012, $\mathcal{B}(\Omega) =$ Bargmann-Fock space and T in the Toeplitz algebra.

Xia and Zheng 2012, If T is an operator on the Bargman-Fock space $\mathcal{F}(\mathbb{C}^n)$ that satisfies

$$|\langle Tk_z, k_w\rangle| \leq \frac{C}{(1+|z-w|)^\beta},$$

for $\beta > 2n$ then $\lim_{z\to\infty} \langle Tk_z, k_z\rangle = 0$ implies that T must be compact.

Xia and Zheng 2012, If T is an operator on the Bargman-Fock space $\mathcal{F}(\mathbb{C}^n)$ that satisfies

$$|\langle Tk_z, k_w\rangle| \leq \frac{C}{(1+|z-w|)^{\beta}},$$

for $\beta > 2n$ then $\lim_{z\to\infty} \langle Tk_z, k_z\rangle = 0$ implies that T must be compact.

Theorem

Let $T : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ be a linear operator. If

$$\lim_{r\to\infty} \sup_{z\in\Omega} \int_{D(z,r)^c} |\langle Tk_z, k_w\rangle| \, d\lambda(w) = 0,$$

and the dual relation holds, then

(a) $\|T\|_e \simeq \sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|U_z^* T U_z f\|$.

(b) *If* $\lim_{z\to\infty} \|Tk_z\| = 0$ *then T must be compact.*

Xia and Zheng 2012, If T is an operator on the Bargman-Fock space $\mathcal{F}(\mathbb{C}^n)$ that satisfies

$$|\langle Tk_z, k_w \rangle| \leq \frac{C}{(1+|z-w|)^\beta},$$

for $\beta > 2n$ then $\lim_{z\to\infty} \langle Tk_z, k_z \rangle = 0$ implies that T must be compact.

Theorem

Let $T : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ be a linear operator. If

$$\lim_{r\to\infty} \sup_{z\in\Omega} \int_{D(z,r)^c} |\langle Tk_z, k_w \rangle| \, d\lambda(w) = 0,$$

and the dual relation holds, then

(a) $\|T\|_e \simeq \sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|U_z^* T U_z f\|$.

(b) *If* $\lim_{z\to\infty} \|Tk_z\| = 0$ *then T must be compact.*

Axler-Zheng argument implies that in the classical spaces:

$$\langle Tk_z, k_z \rangle \to 0 \implies \|Tk_z\| \to 0$$

Localization property

Recall
$$f = \int_\Omega \langle f, k_z \rangle \, k_z d\lambda(z).$$

Localization property

Recall

$$f = \int_{\Omega} \langle f, k_z \rangle \, k_z d\lambda(z).$$

For every $\epsilon > 0$ there exists $r > 0$ large enough such that

$$\|f - \int_{\Omega} \langle f, k_z \rangle \, 1_{D(z,r)} k_z d\lambda(z)\| < \epsilon.$$

Moreover, if T satisfies our conditions, then

$$\|Tf - \int_{\Omega} \langle f, k_z \rangle \, 1_{D(z,r)} Tk_z d\lambda(z)\| < \epsilon.$$

Geometric lemma

Lemma (Whitney Decompositions)

There is a positive integer $N = N(n)$ such that for any $r > 0$ there is a covering of Ω by Borel sets $\{F_j\}$ that satisfy:

(i) $F_j \cap F_k = \emptyset$ *if* $j \neq k$;

(ii) *Every point of* Ω *is contained in at most N sets* $F_j(r) = \{z : d(z, F_j) \leq r\}$;

(iii) *There is a constant* $C(r) > 0$ *such that* $\operatorname{diam}_d F_j \leq C(r)$ *for all* j.

Localization property

As a consequence of the localization property we can prove that

Theorem

Let T be an operator on $\mathcal{B}(\Omega)$ such that

$$\lim_{r\to\infty} \sup_{z\in\Omega} \int_{D(z,r)^c} |\langle Tk_z, k_w\rangle| \, d\lambda(w) = 0,$$

and the dual relation holds. For every $\epsilon > 0$ there exists $r > 0$ and a decomposition $\mathcal{F}_r = \{F_j\}$ of Ω such that

$$\| T - \sum_j M_{1_{F_j}} TP M_{1_{F_j(r)}} \| < \epsilon.$$

Important point:
All finite partial sums of $\sum_j M_{1_{F_j}} TP M_{1_{F_j(r)}}$ are compact.

The essential norm

Estimating the tail we obtain

Theorem

Let T be an operator from the Toeplitz algebra of $\mathcal{B}(\Omega)$. Then

$$\|T\|_{ess} \simeq \sup_{\|f\|\leq 1} \limsup_{z\to\partial\Omega} \|TU_z f\|.$$

Toeplitz operators T on the Bergman space do NOT satisfy:

$$\lim_{r\to\infty} \sup_{z\in\Omega} \int_{D(z,r)^c} |\langle Tk_z, k_w\rangle| \, d\lambda(w) = 0,$$

It fails for $T = I$

Toeplitz operators T on the Bergman space do NOT satisfy:

$$\lim_{r\to\infty}\sup_{z\in\Omega}\int_{D(z,r)^c}|\langle Tk_z,k_w\rangle|\,d\lambda(w)=0,$$

It fails for $T=I$
Uniform Forelli-Rudin estimates:

$$\lim_{r\to\infty}\sup_{z\in\mathbb{B}_n}\int_{D(z,r)^c}|\langle k_z,k_w\rangle|\,\frac{\|K_z\|^a}{\|K_w\|^a}\,d\lambda(w)=0,$$

for $\frac{n-1}{n+1}<a<\frac{n}{n+1}$

Toeplitz operators T on the Bergman space do NOT satisfy:

$$\lim_{r\to\infty}\sup_{z\in\Omega}\int_{D(z,r)^c}|\langle Tk_z,k_w\rangle|\,d\lambda(w)=0,$$

It fails for $T=I$
Uniform Forelli-Rudin estimates:

$$\lim_{r\to\infty}\sup_{z\in\mathbb{B}_n}\int_{D(z,r)^c}|\langle k_z,k_w\rangle|\,\frac{\|K_z\|^a}{\|K_w\|^a}\,d\lambda(w)=0,$$

for $\frac{n-1}{n+1}<a<\frac{n}{n+1}$
Toeplitz operators T on the Bergman space do satisfy:

$$\lim_{r\to\infty}\sup_{z\in\mathbb{B}_n}\int_{D(z,r)^c}|\langle Tk_z,k_w\rangle|\,\frac{\|K_z\|^a}{\|K_w\|^a}\,d\lambda(w)=0,$$

for $\frac{n-1}{n+1}<a<\frac{n}{n+1}$

Theorem

Let T be a linear operator on the Bergman space. Assume

$$\lim_{r\to\infty} \sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} |\langle Tk_z, k_w\rangle| \frac{\|K_z\|^a}{\|K_w\|^a}\, d\lambda(w) = 0,$$

and assume that the dual relation holds.
Then $\lim_{z\to\infty} \|Tk_z\| = 0$ *implies that T must be compact.*

Thank you.