Oscillation of Hölder continuous functions

José González Llorente, Artur Nicolau

The Hölder Class

For $0 < \alpha < 1$,

$$\Lambda_\alpha(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : |f(x) - f(y)| < C|x - y|^\alpha, x, y \in \mathbb{R}\}$$

$\|f\|_\alpha = \inf C$

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Examples. Weierstrass-Hardy Lacunary series.

Given $0 < \alpha < 1$ and $b > 1$, consider

$$f_{\alpha,b}(x) = \sum_{j=0}^{\infty} b^{-j\alpha} \cos(b^j x), \qquad x \in \mathbb{R}$$

Theorem (Hardy, 1916)

- *For any $0 < \alpha \leq 1$ and $b > 1$, $f_{\alpha,b}$ is nowhere differentiable*
- *If $0 < \alpha < 1$, $f_{\alpha,b} \in \Lambda_\alpha(\mathbb{R})$ and for any $x \in \mathbb{R}$,*

$$\limsup_{h \to 0} \frac{|f_{\alpha,b}(x+h) - f_{\alpha,b}(x)|}{|h|^\alpha} > 0$$

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Extreme case $\alpha = 1$

$$f_{1,b} \in \Lambda_*(\mathbb{R}) = \{f : \sup_{x,h\in\mathbb{R}} \frac{|f(x+h)+f(x-h)-2f(x)|}{|h|} < \infty\}$$

At a.e. $x \in \mathbb{R}$,

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Fix $0 < \alpha < 1$, $b > 1$. For almost every $x \in \mathbb{R}$,

$$h \to \frac{f_{\alpha,b}(x+h) - f_{\alpha,b}(x)}{|h|^\alpha}$$

oscillates between $-C(\alpha, b)$ and $C(\alpha, b)$ when $h \to 0$

Random Walk

Given a function $f \in \mathrm{Lip}_\alpha(\mathbb{R})$ and $0 < \varepsilon < 1/2$, consider

$$\Theta_\varepsilon(f)(x) = \int_\varepsilon^1 \frac{f(x+h) - f(x-h)}{h^\alpha} \frac{dh}{h}, \quad x \in \mathbb{R}$$

- Trivial estimate: $\|\Theta_\varepsilon(f)\|_\infty \leq 2^\alpha \|f\|_\alpha \log 1/\varepsilon$
- Best possible: For $f(x) = |x|^\alpha \operatorname{sign}(x)$, $\Theta_\varepsilon(f)(0) \geq C \log 1/\varepsilon$

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Theorem

$0 < \alpha < 1$ and $f \in \Lambda_\alpha(\mathbb{R})$

- *For any interval $I \subset \mathbb{R}$, $|I| = 1$,*

$$\int_I |\Theta_\varepsilon(f)(x)|^2 \, dx \le c(\alpha)(\log 1/\varepsilon)\|f\|_\alpha^2$$

- *At almost every point $x \in \mathbb{R}$, one has*

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Sharpness

- For
$$f_{\alpha,2} = \sum_j 2^{-j\alpha} \sin(2\pi 2^j t) \in \Lambda_\alpha(\mathbb{R})$$
one has the converse estimates
- For b large enough,
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Related LIL's

Theorem (Makarov,1984)

$f : \mathbb{D} \to \mathbb{C}$ analytic, $\|f\|_B = \sup\{(1-|z|)|f'(z)| : z \in \mathbb{D}\} < \infty$. Then

$$\limsup_{r\to 1} \frac{|f(re^{it})|}{\sqrt{\log 1/(1-r) \log\log\log 1/(1-r)}} \leq C\|f\|_B \text{, a.e.t}$$

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Theorem (Lyubarskii-Malinnikova and Solberg-Malinnikova-Mozolyako, 2012)

u harmonic in $\mathbb{D}$, $|u(z)| \leq V(|z|)$. Consider

$$I(f)(re^{it}) = \int_0^r u(se^{it})d(\frac{1}{V(s)})$$

Then,

$$\limsup_{r\to 1} \frac{|I(f)(re^{it})|}{\sqrt{\log V(r) \log\log\log V(r)}} \leq C\,, a.e.t$$

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Subgaussian Estimate

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Subgaussian Estimate:

$$|\{x \in [0,1] : |\Theta_\varepsilon^*(f)(x)| > t\sqrt{\log 1/\varepsilon}| \le ce^{-t^2/c}.$$

Here $\Theta_\varepsilon^*(f)(x) = \sup\{|\Theta_\delta(f)(x)| : 1/2 \ge \delta \ge \varepsilon\}$

- Gives L^p estimates
- Gives LIL

Exponential Inequality: For any $\lambda > 0$,

$$\int_0^1 \exp\left(\lambda\Theta_\varepsilon^*(f)(x)\right) dx \le c\exp\left(c(\alpha)\lambda^2\log(1/\varepsilon)\right)$$

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Dyadic Model

- $\mathcal{D}_k$: dyadic intervals of length 2^{-k}
- A sequence of funcions $\{S_k\}$ is a dyadic martingale if
  (a) S_k is constant in each interval of $\mathcal{D}_k$
  (b) For any $I \in \mathcal{D}_k$, $\int_I (S_{k+1}(x) - S_k(x))\, dx = 0$
- Quadratic Variation: $\langle S \rangle_n^2(x) = \sum_{k=1}^n (S_k(x) - S_{k-1}(x))^2$

Theorem

- $\{x : \lim_{n\to\infty} S_n(x) \textit{ exists}\} = \{x : \langle S \rangle_\infty(x) < \infty\}$ *a.e.*
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- *For $0 < p < \infty$, $\|sup_n |S_n|\|_p \simeq \|\langle S \rangle_\infty\|_p$*

Dyadic Model

- $\mathcal{D}_k$: dyadic intervals of length 2^{-k}
- A sequence of funcions $\{S_k\}$ is a dyadic martingale if
  (a) S_k is constant in each interval of $\mathcal{D}_k$
  (b) For any $I \in \mathcal{D}_k$, $\int_I (S_{k+1}(x) - S_k(x))\,dx = 0$
- Quadratic Variation: $\langle S \rangle_n^2(x) = \sum_{k=1}^n (S_k(x) - S_{k-1}(x))^2$

Theorem

- $\{x : \lim_{n\to\infty} S_n(x)$ *exists*$\}$ = $\{x : \langle S \rangle_\infty(x) < \infty\}$ *a.e.*
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Discrete Exponential Inequality

Assume $S_0 \equiv 0$. Then for any $\lambda > 0$,

$$\int_0^1 \exp(\lambda S_N^*(x))\,dx \leq C \exp\left(C\lambda^2 \|\langle S\rangle_N\|_\infty^2\right)$$

Here $S_N^*(x) = \sup_{n\leq N} |S_n(x)|$

Martingale of Divided Differences

Given $f : \mathbb{R} \to \mathbb{R}$, consider the dyadic martingale

$$S_k(x) = \frac{f(b) - f(a)}{b - a} \quad , x \in I = (a, b) \in \mathcal{D}_k$$

If $f \in \Lambda_\alpha$, then $\|S_k\|_\infty \leq C2^{k(1-\alpha)}$
For $\varepsilon = 2^{-N}$,

$$\Theta_\varepsilon(f)(x) = \int_\varepsilon^1 \frac{f(x+h) - f(x-h)}{h^\alpha} \frac{dh}{h}$$

$$\simeq \sum_{k=1}^{N} \frac{f(b_k(x)) - f(a_k(x))}{2^{-k\alpha}} = \sum_{k=1}^{N} S_k(x) 2^{k(\alpha-1)}$$

where $x \in (a_k(x), b_k(x)) \in \mathcal{D}_k$.
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Discrete Version of the Main Result

For $\beta = 1 - \alpha$, $\Gamma_N(x) = \sum_{k=1}^{N} 2^{-k\beta} S_k(x)$.

Trivial Estimate: $\|\Gamma_N\|_\infty \leq CN$

Theorem

Let $\{S_n\}$ be a dyadic martingale with $S_0 = 0$ and $\|S_n\|_\infty \leq C2^{n\beta}$, $n = 1, 2, \ldots$. Consider $\Gamma_N^*(x) = \sup_{k \leq N} |\Gamma_k(x)|$. Then,

- $\int_0^1 \exp\left(\lambda \Gamma_N^*(x)\right)\, dx \leq Ce^{C\lambda^2 N}, \lambda > 0, N = 1, 2, \ldots,$
- $\int_0^1 |\Gamma_N^*(x)|^2\, dx \leq CN, N = 1, 2, \ldots$
- $\limsup_{N\to\infty} \frac{|\Gamma_N(x)|}{\sqrt{N \log\log N}} \leq C \quad a.e. x \in [0, 1],$

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Proof. Discrete Version

- Γ_N is not a dyadic martingale but

$$T_n := \sum_{k=1}^{n} 2^{-k\beta}(S_k - S_{k-1}) = C(\beta)\Gamma_n + 0(1)$$

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- For $1 \leq \rho \leq 2$, $\mathcal{D}_k(\rho)$: dyadic intervals of length $2^{-k}\rho$
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- Consider $f_s(x) = f(x - s)$. Then

$$\Theta_{2^{-N}}(f)(x) = c \int_1^2 \int_0^1 \Gamma_N^\rho(f_s)(x + s)\, ds \frac{d\rho}{\rho^2} + O(1)$$

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