Cyclic Elements
on the Hilbert Multidisc

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1. Cyclic Elements

- Cyclic elements of a family of operators

Θ= { T } a family of operators on X, $T:X\rightarrow X$

G= Semi-Group(Θ)

DEFINITION: x in X is cyclic if Span(Gx)= X.

CYC(G) – the set of all G-cyclic elements

To know CYC(G) – a step to Lat(G).

2. Example (classical)

The Hardy space $H^2(\mathbb{D})$ of the Disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:

$$H^2(\mathbb{D}) = \{f = \sum_{n \in \mathbb{Z}_+} \hat{f}(n) z^n : \|f\|_2^2 = \sum_{n \in \mathbb{Z}_+} |\hat{f}(n)|^2 < \infty\},$$

T is the shift operator $Tf = zf$, $G = \{T^n\}_{n \in \mathbb{Z}_+}$.

THEOREM (V.I.Smirnov,1932 - A.Beurling,1949):

$f \in H^2$ is cyclic $\Leftrightarrow$ f is "outer": $log|f(0)| = \int_{\mathbb{T}} log|f|dm$.

All known proofs depend on the canonical factorization $f = f_{inn} \cdot f_{out}$.

3. The Framework of this Talk

The Hilbert multi-disc $\mathbb{D}_2^\infty$

- **D. Hilbert (1909)** defined an infinite-dimensional multi-disc $\mathbb{D}_2^\infty$,

$$\mathbb{D}_2^\infty = \{\zeta = (\zeta_k)_{k\geq 1} \in l^2 : |\zeta_k| < 1 \ (\forall k)\},$$

and sketched a holomorphic function theory on $\mathbb{D}_2^\infty$.

- **The Hardy space on $\mathbb{D}_2^\infty$,**

$$H^2(\mathbb{D}_2^\infty) =: \{F = \sum_{\alpha\in\mathbb{Z}_+(\infty)} c_\alpha(F)\zeta^\alpha : \|F\|_2^2 = \sum_{\alpha\in\mathbb{Z}_+(\infty)} |c_\alpha(F)|^2 < \infty\},$$

$\mathbb{Z}_+(\infty) = \cup_{k\geq 1} \mathbb{Z}_+^k$ all finitely supported sequences of nonnegative integers $\alpha = (\alpha_1, ..., \alpha_s, 0, 0, ...)$, and $\zeta^\alpha = \zeta_1^{\alpha_1} ... \zeta_s^{\alpha_s}$ $(\zeta \in \mathbb{D}_2^\infty)$.

3. The Framework of this Talk (cnd)

• **The multiplication (monomial) semigroupe,** $M_\zeta = (\zeta^\alpha)_{\alpha \in \mathbb{Z}_+(\infty)}$,

$$\zeta^\alpha : \; f(\zeta) \longrightarrow \zeta^\alpha f(\zeta), \; \zeta \in \mathbb{D}_2^\infty.$$

• $Lat(M_\zeta)$ **= the lattice of closed** M_ζ**-invariant subspaces of** $H^2(\mathbb{D}_2^\infty)$.

• $F \in H^2(\mathbb{D}_2^\infty)$ **is** M_ζ**-cyclic iff** $Span(M_\zeta F) = H^2(\mathbb{D}_2^\infty)$.

• **EXAMPLES:**

1) (trivial) $F = \mathbb{1} \in CYC(M_\zeta)$; **2) (obvious)** $F \in CYC(M_\zeta)$ **if** $F \in H^2(\mathbb{D}_2^\infty)$ **and** $1/F \in H^\infty(\mathbb{D}_2^\infty)$,

3) (simple) $F(\zeta) = exp(\frac{\zeta_1+1}{\zeta_1-1})$ **is not cyclic.**

4. The Problem

- **Describe** $CYC(M_\zeta)$.
- **Describe** $Lat(M_\zeta)$.

Remark: clearly, $F \in CYC(M_\zeta) \Leftrightarrow (F \notin E\ \forall E \in Lat(M_\zeta),\ E \neq H^2(\mathbb{D}_2^\infty))$.

5. First Observations

(1) $\forall \lambda \in \mathbb{D}_2^\infty$

$$F \longmapsto F(\lambda) \text{ is BDD on } H^2(\mathbb{D}_2^\infty)$$

(2) The reproducing kernel of $H^2(\mathbb{D}_2^\infty)$ is an "Euler product"

$$k_\lambda(\zeta) = \sum_{\alpha \geq 0} \overline{\lambda}^\alpha \zeta^\alpha = \prod_{j \geq 1} \frac{1}{1 - \overline{\lambda}_j \zeta_j}, \ \|k_\lambda\|_{H^2}^2 = \prod_{j \geq 1} \frac{1}{1 - |\lambda_j|^2} < \infty.$$

Remark: $k_\lambda \in H^\infty(\mathbb{D}_2^\infty) \Leftrightarrow \lambda \in l^1$.

(3) $H^2(\mathbb{D}^m) \subset H^2(\mathbb{D}_2^\infty)$ (isometrically), and $F \in H^2(\mathbb{D}^m)$ is cyclic in $H^2(\mathbb{D}^m) \Leftrightarrow F \in CYC(M_\zeta)$.

(4) Similar is true for $H^2_\sigma(\mathbb{D}_2^\infty) = \{F \in H^2(\mathbb{D}_2^\infty) :$ *Fourier spectrum of F is in σ*$\}$ for every "half-group" $\sigma \subset \mathbb{Z}_+(\infty)$. EXAMPLE: $\sigma = \alpha \cdot \mathbb{Z}_+$, where $\alpha \in \mathbb{Z}_+(\infty)$.

6. Results

THEOREM 1: IF $(\exists\epsilon > 0$ **s.t.** $F^{1+\epsilon} \in H^2(\mathbb{D}_2^\infty),\ 1/F^\epsilon \in H^2(\mathbb{D}_2^\infty))$ **THEN** $F \in CYC(M_\zeta)$.

Proof. - SOME PROPERTIES OF $H^p(\mathbb{D}_2^\infty)$ SPACES FOR $p \geq 2$ WILL BE USED, IN PARTICULAR $H^p(\mathbb{D}_2^\infty) \subset H^q(\mathbb{D}_2^\infty)$ FOR $p > q$, AND POLYNOMIALS IN ζ^α, $\alpha \in \mathbb{Z}_+(\infty)$ ARE DENSE IN $H^p(\mathbb{D}_2^\infty)$.

WLOG $\epsilon = 1/N$, N entire; let $\gamma = \frac{\epsilon}{N(1+\epsilon)}$, $q = \frac{2(1+\epsilon)}{\epsilon}$.
THEN $1/F^\gamma \in H^q$ and $\exists$ *poly* p_k s.t. $lim_k \|F^{-\gamma} - p_k\|_q = 0$. By Hölder

$$\|F^{1-\gamma} - p_k F\|_2 = \|F(\frac{1}{F^\gamma} - p_k)\|_2 \leq \|F^{1+\epsilon}\|_2^{1/1+\epsilon} \|\frac{1}{F^\gamma} - p_k\|_q \longrightarrow 0.$$

HENCE $F^{1-\gamma} \in E =: Span_{H^2}(M_\zeta F)$. NEXT, $F^{1-2\gamma} \in E$, etc., by induction $1 \in E$.
CONCLUSION: $E = H^2(\mathbb{D}_2^\infty)$. ∎

• **REMARK - a necessary condition:** $F \in CYC(M_\zeta) \;\Rightarrow\; F(\zeta) \neq 0 \;\forall \zeta \in \mathbb{D}_2^\infty$.

THEOREM 2: IF $F \in Hol((1+\epsilon)\mathbb{D}^m)$ **AND** $F(\zeta) \neq 0 \;\forall \zeta \in \mathbb{D}_2^\infty$ **THEN** $F \in CYC(M_\zeta)$.

Proof. - We will check conditions of Theorem 1: $F \in H^p(\mathbb{D}_2^\infty)$ is obvious $(\forall p)$.

LEMMA (the zero set of F**): let** $Z(F) =: \{\zeta \in \overline{\mathbb{D}}^m : \; F(\zeta) = 0\}$**, then** $\exists\; \sigma \subset \{1, 2, ..., m\}$ **s.t.**

$$Z(F) = A \times \mathbb{D}^\sigma,\; A \subset \mathbb{T}^{\sigma'}$$

where $\sigma' = \{1, 2, ..., m\} \backslash \sigma$ **and** A **is a finite union of analytic manifolds of real dimensions strictly less than** $card(\sigma')$**.**

Proof of Theorem 2 (cnd)

Applying S.Lojaciewicz's theorem for F: $\exists N,\ C_1 > 0$ s.t.

$$|F(\zeta)| \geq C_1(dist(\zeta, Z(F)))^N = C_1 1(dist(\zeta_{\sigma'}, A))^N$$

for every $\zeta \in \mathbb{T}^m$, and $\exists C_2 > 0$ s.t. $dist(r\zeta_{\sigma'}, A) \geq C_2 \cdot dist(\zeta_{\sigma'}, A)$ $(0 < r < 1,\ \zeta_{\sigma'} \in \mathbb{T}^{\sigma'})$. Let $d = card(\sigma')$, $\epsilon > 0$ s.t. $N\epsilon < 1$, then

$$\int_{\mathbb{T}^m} \frac{d\Lambda_m(\zeta)}{|F(r\zeta)|^\epsilon} \leq C_3 \int_{\mathbb{T}^{\sigma'}} \frac{d\Lambda_d(\zeta_{\sigma'})}{(dist(\zeta_{\sigma'}, A))^{N\epsilon}} < \infty,$$

since A is C^∞ diffeomorphic to $H = \{x = (x_1, ..., x_d) :\ x_1 = 0\} \subset \mathbb{R}^d$, and with $\zeta_{\sigma'} \sim x = (x_1, ..., x_d)$, $dist(\zeta_{\sigma'}, A) \geq c|x_1|$,

$$c \cdot \int_{|x_j|<1, \forall j} \frac{dx_1 ... dx_d}{|x_1|^{N\epsilon}} < \infty.$$

It follows that $1/F \in H^\epsilon(\mathbb{D}^m)$. By Theorem 1, F is cyclic. ∎

7. Two Corollaries

COROLLARY 1: Reproducing kernels k_λ, $\lambda \in \mathbb{D}_2^\infty$, are cyclic.

Indeed,

- (obviously) $k_\lambda(\zeta) = \prod_{s\geq 1} F_{\lambda_s}(\zeta_s)$ where $F_a(z) = (1-\overline{a}z)^{-1}$ $(a, z \in \mathbb{D})$;
- $\|F_a\|^p_{H^p(\mathbb{T})} = 1 + |pa/2|^2(1+o(1))$ as $a \longrightarrow 0$ $(\forall p < \infty)$, and hence
- $k_\lambda,\ 1/k_\lambda \in H^p(\mathbb{D}_2^\infty)$ for every λ, $\lambda \in \mathbb{D}_2^\infty$ and $\forall p < \infty$. ∎

COROLLARY 2: $F \in H^2(\mathbb{D}_2^\infty)$, $ReF(\zeta) \geq 0$ $(\zeta \in \mathbb{D}_2^\infty) \Rightarrow$ F is cyclic. In particular, $F = 1 + f$, $\|f\|_\infty \leq 1$, is cyclic. ∎

Rem: Corollaries are equivalent to Hedenmalm, Lindquist, and Seip's results (1997) (here with new and easier proofs).

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8. Why it is important, tenthly?

(10) Because it is equivalent to the dilation f (nx), n=1,2,… completeness problem (DCP)

(11) Beacuse a partial case of the DCP is equivalent to the Riemann hypothesis

9. What is the DCP?

DILATION COMPLETENESS PROBLEM: *To describe functions $f \in L^p(0,\infty)$ such that*

$$span_{L^p(0,1)}(f(nx): \ n = 1, 2, 3, ...) = \ L^p(0, 1).$$

EXAMPLES:

(1) Functions with COMPLETE dilations: $f = \ Sin(\pi x)$; $f = e^{-x}$; $f = \ e^{-x^\alpha}$, $0 < \alpha \leq 1$;

(2) Functions with INCOMPLETE dilations: $f = Sin(2\pi x)$; $f = e^{-x^\alpha}$, $\alpha > 1$;

(3) Riemann Hypothesis is equivalent to the DCP for $f(x) = \frac{1}{x} - [\frac{1}{x}]$, $x > 0$ **(B.Nyman 1950; L.Báez-Duarte 2003).**

10. Periodic DCP

The FIRST STEP TO THE DCP - a PERIODIC DCP raised by Wintner (1944) and Beurling (1945), i.e. the question when $f(nx)$, $n = 1, 2, \ldots$ are complete in $L^2(0, 1)$ if

$$f \text{ is odd and 2-periodic.}$$

- **Aurel Wintner, 1944**, in Amer. J. Math., motivated by analytical problems arising from the Eratosthenes sieve method.

- **Arne Beurling, 1945**, in a seminar talk at Uppsala University, whithout a declared motivation.

10. Periodic DCP (cnd)

- **The DILATIONS on $L^2_{odd}(\mathbb{R}/2\mathbb{Z})$ FORM an OPERATOR SEMIGROUP:**

$$f \in L^2_{odd}(-1,1) \Rightarrow f = \sum_{k\geq 1} b_k Sin(\pi k x), \ \sum_{k\geq 1} |b_k|^2 < \infty.$$

HENCE $f(nx) = (T_n f)(x)$, **where** (T_n) **acts on an ONB** $e_k = Sin(\pi k x)$ **as** $T_n e_k = e_{nk}$.

- **CHANGING the BASIS:** $(e^{ikx})_{k\geq 1}$ **in the Hardy space** $H^2_0(\mathbb{D})$**,**

$$H^2_0(\mathbb{D}) = \{f = \sum_{k\geq 1} a_k z^k : \ \|f\|_2^2 = \sum_{k\geq 1} |a_k|^2 < \infty\},$$

we get a semigroup of isometries

$$T_n f(z) = f(z^n), \ f \in H^2_0(\mathbb{D}).$$

- **Now, the Periodic DCP is TO FIND CYCLIC VECTORS** f **of** (T_n)**:**

$$span_{H^2}(T_n f : \ n \geq 1) = H^2_0(\mathbb{D}).$$

10. Periodic DCP (cnd)

A unitary equivalence between $H_0^2(\mathbb{D})$ and $H^2(\mathbb{D}_2^\infty)$:

$$U: \ f = \sum_{n\geq 1} \hat{f}(n) z^n \ \longmapsto \ Uf(\zeta) = \sum_{n\geq 1} \hat{f}(n)\zeta^{\alpha(n)}, \ \zeta \in \mathbb{D}_2^\infty,$$

$\alpha(n) = (\alpha_1, .., \alpha_s, 0, ...)$ **is defined by the prime decomposition**

$$n = p_1^{\alpha_1} ... p_s^{\alpha_s}, \ \alpha_j \in \mathbb{Z}_+.$$

LEMMA. (1) *U is unitary $H_0^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D}_2^\infty)$ and transforms (T_n) into a multiplication semigroup $M_\zeta = (M_{\zeta^\alpha})_{\alpha\in\mathbb{Z}_+(\infty)}$:*

$$(UT_nU^{-1})f(\zeta) = \zeta^{\alpha(n)} f(\zeta) \ (\zeta \in \mathbb{D}_2^\infty, \ f \in H^2(\mathbb{D}_2^\infty)).$$

(2) *$E \in Lat(T_n) \Leftrightarrow UE \in Lat(M_\zeta)$; a function $f \in H_0^2$ is (T_n)-cyclic iff Uf is M_ζ-cyclic.*

10. Periodic DCP (cnd)

- Conclusion: the following three problems are now equivalent:

(1) $M\zeta$ cyclicity on the Hilbert multidisc

(2) Completeness of $f(z^n)$, n=1,2,… in $H^2(D)$

(3) Completeness of $g(nx)$, n=1,2,... In $L^2(0,1)$, with 2-periodic odd extention of g

10. Periodic DCP (cnd)

An Abridged History of the Periodic DCP

- **A.Wintner, 1944:** $f = \sum_{n\geq 1} n^{-s} z^n$, $Re(s) > 1/2$ is (T_n)-cyclic.

- **A.Beurling, 1945:** f is (T_n)-cyclic $\Rightarrow Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$.

- **R.Gosselin and J.Neuwirth, 1968,**
- **J.Ginsberg, J.Neuwirth, and D.Newman, 1970:** f a polynomial, then f cyclic $\Leftrightarrow Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$.
- **H.Hedenmalm, P.Lindquist, and K.Seip, 1997-1998:** $f = z + \sum_{n\geq 2} a_n z^n$, $\sum_{n\geq 2} |a_n| \leq 1 \Rightarrow f$ is cyclic; $f = \Sigma_{n\geq 1} \chi(n) z^n$ is cyclic, where χ stands for a bounded character of $\mathbb{N}$.

Remark: As it is shown all these results follow from Theorem 1 above.

11. Two Open Questions

(1) Is Theorem 1 sharp?

Whether there exists a NON-CYCLIC $F \in H^2(\mathbb{D}_2^\infty)$ such that $1/F \in H^2(\mathbb{D}_2^\infty)$?

COMMENTS:

- $F \in H^2(\mathbb{D})$, $1/F \in H^2(\mathbb{D})$ **(obviously) implies that** F **is CYCLIC.**
- **It is plausible that for** $H^2(\mathbb{D}^2)$ **there is a counterexample:**

HINT: if $f(0) = 0$ **and** $\varphi =: f' \in L_a^2(\mathbb{D})$ **(the Bergman space on** $\mathbb{D}$**) then**

a) $\varphi \longmapsto \dfrac{f(w) - f(z)}{w - z} =: F \in H^2(\mathbb{D}^2)$ **(isometric embedding** $L_a^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D}^2)$**),**

b) if F **is cyclic in** $H^2(\mathbb{D}^2)$**, then** φ **is cyclic in** $L_a^2(\mathbb{D})$**,**

c) there exists $\varphi \in L_a^2(\mathbb{D})$, $1/\varphi \in L_a^2(\mathbb{D})$ **which is NOT CYCLIC (A.Borichev and H.Hedenmalm)**

11. Two Open Questions (cnd)

(2) V.Ya. Kozlov's problem (1950):

Let $0 < \beta < 1$ **and** $f_\beta \in L^2_{odd}(-1,1)$, $f_\beta|(0,1) = \chi_{(0,\beta)}$, **2-periodically extended on** $\mathbb{R}$**. For which values of** β

$$f_\beta(nx),\ n = 1, 2, ...$$

are complete in $L^2_{odd}(-1,1)$**?**

Known cases (claimed V.Ya.Kozlov, Dokl. URSS, 73(1950), 441-444, no proofs):

CYCLIC:

1) $\beta = 1$ **(equivalent to reproducing kernel** k_λ**,** $\lambda = (0, 1/p_2, 1/p_3, ...)$**)**

2) $\beta = 1/2,\ 2/3$**;**

NON-CYCLIC:

3) $\beta = 1/3$ **(and a neighborhood);**

4) $\beta = q/p$**,** p **prime** $p \neq 2$**,** q **odd, and** $tan^2(2\pi\beta) < 1/p$**.**

THE END

THANK YOU!