Hardy spaces integration operators

Jordi Pau

Integration operators between Hardy spaces in the unit ball

Jordi Pau

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona

Hilbert Function Spaces
Gargnano, May 2013

Integration operators

Let $g \in H(\mathbb{D})$. Define the linear operator

$$J_g f(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta, \qquad f \in H(\mathbb{D}).$$

The operator J_g includes (as a particular cases):

1. the Volterra operator (choose $g(z) = z$).
2. the Cesàro operator (choose $g(z) = \log(1 - z)$).

Integration operators

Let $g \in H(\mathbb{D})$. Define the linear operator

$$J_g f(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta, \qquad f \in H(\mathbb{D}).$$

The operator J_g includes (as a particular cases):

1. the Volterra operator (choose $g(z) = z$).
2. the Cesàro operator (choose $g(z) = \log(1 - z)$).

Hardy spaces in the unit disk

Theorem [Aleman-Siskakis: $p \geq 1$; Aleman-Cima: $0 < p < 1$]

Let $0 < p < \infty$ and $g \in H(\mathbb{D})$. Then J_g is bounded on $H^p(\mathbb{D})$ if and only if $g \in BMOA$.

Theorem [Aleman-Cima; 2001]

Let $0 < p < q < \infty$ and $g \in H(\mathbb{D})$. Let $\alpha = \frac{1}{p} - \frac{1}{q}$.

- If $\alpha \leq 1$, then $J_g : H^p(\mathbb{D}) \to H^q(\mathbb{D})$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\alpha} |g'(z)| < \infty.$$

- If $\alpha > 1$ then $J_g : H^p(\mathbb{D}) \to H^q(\mathbb{D})$ is bounded if and only if g is constant.

Hardy spaces in the unit disk

Theorem [Aleman-Cima; 2001]

Let $0 < q < p < \infty$. Then $J_g : H^p(\mathbb{D}) \to H^q(\mathbb{D})$ is bounded if and only if $g \in H^r$ with $r = \dfrac{pq}{p-q}$.

The proof uses in a decisive way the strong factorization for functions in the Hardy space:

$$H^p = H^s \cdot H^t, \qquad \frac{1}{p} = \frac{1}{s} + \frac{1}{t}.$$

Theorem [Aleman-Siskakis; 1995]

(i) If $1 < p < \infty$ then $J_g \in S_p(H^2)$ if and only if $g \in B_p$.

(ii) If $0 < p \leq 1$ then $J_g \in S_p(H^2)$ if and only if g is constant.

Hardy spaces in the unit disk

Theorem [Aleman-Cima; 2001]

Let $0 < q < p < \infty$. Then $J_g : H^p(\mathbb{D}) \to H^q(\mathbb{D})$ is bounded if and only if $g \in H^r$ with $r = \dfrac{pq}{p-q}$.

The proof uses in a decisive way the strong factorization for functions in the Hardy space:

$$H^p = H^s \cdot H^t, \qquad \frac{1}{p} = \frac{1}{s} + \frac{1}{t}.$$

Theorem [Aleman-Siskakis; 1995]

(i) If $1 < p < \infty$ then $J_g \in S_p(H^2)$ if and only if $g \in B_p$.

(ii) If $0 < p \leq 1$ then $J_g \in S_p(H^2)$ if and only if g is constant.

Hardy spaces in the unit disk

Theorem [Aleman-Cima; 2001]

Let $0 < q < p < \infty$. Then $J_g : H^p(\mathbb{D}) \to H^q(\mathbb{D})$ is bounded if and only if $g \in H^r$ with $r = \dfrac{pq}{p-q}$.

The proof uses in a decisive way the strong factorization for functions in the Hardy space:

$$H^p = H^s \cdot H^t, \qquad \frac{1}{p} = \frac{1}{s} + \frac{1}{t}.$$

Theorem [Aleman-Siskakis; 1995]

(i) If $1 < p < \infty$ then $J_g \in S_p(H^2)$ if and only if $g \in B_p$.

(ii) If $0 < p \leq 1$ then $J_g \in S_p(H^2)$ if and only if g is constant.

Integration operators in the unit ball

Hardy spaces integration operators

Jordi Pau

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$. For $g \in H(\mathbb{B}_n)$ define the linear operator

$$J_g f(z) = \int_0^1 f(tz)\, Rg(tz)\, \frac{dt}{t}, \qquad z \in \mathbb{B}_n$$

for f holomorphic in $\mathbb{B}_n$.

Rg is the radial derivative of g

$$Rg(z) = \sum_{k=1}^{n} z_k \frac{\partial g}{\partial z_k}(z), \qquad z = (z_1, \dots, z_n) \in \mathbb{B}_n.$$

Basic property:

$$R(J_g f)(z) = f(z)\, Rg(z).$$

Integration operators in the unit ball

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$. For $g \in H(\mathbb{B}_n)$ define the linear operator

$$J_g f(z) = \int_0^1 f(tz)\, Rg(tz)\, \frac{dt}{t}, \qquad z \in \mathbb{B}_n$$

for f holomorphic in $\mathbb{B}_n$.

Rg is the radial derivative of g

$$Rg(z) = \sum_{k=1}^{n} z_k \frac{\partial g}{\partial z_k}(z), \qquad z = (z_1, \dots, z_n) \in \mathbb{B}_n.$$

Basic property:

$$R(J_g f)(z) = f(z)\, Rg(z).$$

Integration operators in the unit ball

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$. For $g \in H(\mathbb{B}_n)$ define the linear operator

$$J_g f(z) = \int_0^1 f(tz)\, Rg(tz)\, \frac{dt}{t}, \qquad z \in \mathbb{B}_n$$

for f holomorphic in $\mathbb{B}_n$.

Rg is the radial derivative of g

$$Rg(z) = \sum_{k=1}^{n} z_k \frac{\partial g}{\partial z_k}(z), \qquad z = (z_1, \ldots, z_n) \in \mathbb{B}_n.$$

Basic property:

$$R(J_g f)(z) = f(z)\, Rg(z).$$

Hardy spaces on the unit ball

For $0 < p < \infty$, the Hardy space $H^p(\mathbb{B}_n)$ consists of those functions $f \in H(\mathbb{B}_n)$ with

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_{\mathbb{S}_n} |f(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p} < \infty.$$

Theorem [Pau; 2012]

Let $g \in H(\mathbb{B}_n)$ and $0 < p < \infty$. Then J_g is bounded on $H^p(\mathbb{B}_n)$ if and only if $g \in BMOA(\mathbb{B}_n)$. Moreover,

$$\|J_g\| \asymp \|g\|_{BMOA}.$$

Hardy spaces on the unit ball

Hardy spaces integration operators

Jordi Pau

For $0 < p < \infty$, the Hardy space $H^p(\mathbb{B}_n)$ consists of those functions $f \in H(\mathbb{B}_n)$ with

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_{\mathbb{S}_n} |f(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p} < \infty.$$

Theorem [Pau; 2012]

Let $g \in H(\mathbb{B}_n)$ and $0 < p < \infty$. Then J_g is bounded on $H^p(\mathbb{B}_n)$ if and only if $g \in BMOA(\mathbb{B}_n)$. Moreover,

$$\|J_g\| \asymp \|g\|_{BMOA}.$$

Boundedness on Hardy spaces

Let $0 < \alpha \leq 1$. A function $g \in H(\mathbb{B}_n)$ belongs to $\Lambda(\alpha)$ if

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{1-\alpha} |Rg(z)| < \infty.$$

Theorem [Pau; 2012]

Let $g \in H(\mathbb{B}_n)$ and $0 < p < q < \infty$. Let $\alpha = n(\frac{1}{p} - \frac{1}{q})$.

(i) If $\alpha \leq 1$ then $J_g : H^p(\mathbb{B}_n) \to H^q(\mathbb{B}_n)$ is bounded if and only if $g \in \Lambda(\alpha)$.

(ii) If $\alpha > 1$ then $J_g : H^p(\mathbb{B}_n) \to H^q(\mathbb{B}_n)$ is bounded if and only if g is constant.

This was previously obtained by Avetysan-Stevic in 2009.

Boundedness on Hardy spaces

Theorem [Pau; 2013]

Let $g \in H(\mathbb{B}_n)$ and $0 < q < p < \infty$. Then $J_g : H^p(\mathbb{B}_n) \to H^q(\mathbb{B}_n)$ is bounded if and only if $g \in H^r(\mathbb{B}_n)$ with $r = pq/(p-q)$. Moreover,

$$\|J_g\|_{H^p \to H^q} \asymp \|g\|_{H^r}.$$

Membership in Schatten p-classes

Theorem [Pau; 2013]

Let $g \in H(\mathbb{B}_n)$. Then

(a) For $n < p < \infty$, J_g belongs to $S_p(H^2)$ if and only if $g \in B_p$, that is,

$$\int_{\mathbb{B}_n} |Rg(z)|^p \, (1 - |z|^2)^p \, d\lambda_n(z) < \infty.$$

(b) If $0 < p \leq n$ then J_g is in $S_p(H^2)$ if and only if g is constant.

Here

$$d\lambda_n(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}.$$

Hardy-Stein type inequalities

If $g(0) = 0$ then

$$\|g\|_{H^p}^p \asymp \int_{\mathbb{B}_n} |g(z)|^{p-2} |Rg(z)|^2 \,(1 - |z|^2) dv(z)$$

There is also a version using the invariant gradient

$$\widetilde{\nabla} f(z) = \nabla (f \circ \varphi_z)(0)$$

$$\|g\|_{H^p}^p \asymp \int_{\mathbb{B}_n} |g(z)|^{p-2} |\widetilde{\nabla} g(z)|^2 \,(1 - |z|^2)^n d\lambda_n(z)$$

The admissible maximal function

For $\alpha > 1$, the admissible approach region at $\zeta \in \mathbb{S}_n$ is

$$\Gamma(\zeta) = \Gamma_\alpha(\zeta) = \{z \in \mathbb{B}_n : |1 - \langle z, w\rangle| < \frac{\alpha}{2}(1 - |z|^2)\}$$

The admissible maximal function is

$$f^*(\zeta) = f^*_\alpha(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|, \qquad \zeta \in \mathbb{S}_n$$

- Let $0 < p < \infty$ and $f \in H(\mathbb{B}_n)$. Then

$$\|f^*\|_{L^p(\mathbb{S}_n)} \leq C\,\|f\|_{H^p}.$$

The admissible maximal function

For $\alpha > 1$, the admissible approach region at $\zeta \in \mathbb{S}_n$ is

$$\Gamma(\zeta) = \Gamma_\alpha(\zeta) = \{z \in \mathbb{B}_n : |1 - \langle z, w\rangle| < \frac{\alpha}{2}(1 - |z|^2)\}$$

The admissible maximal function is

$$f^*(\zeta) = f^*_\alpha(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|, \qquad \zeta \in \mathbb{S}_n$$

- Let $0 < p < \infty$ and $f \in H(\mathbb{B}_n)$. Then

$$\|f^*\|_{L^p(\mathbb{S}_n)} \leq C\,\|f\|_{H^p}.$$

The admissible maximal function

For $\alpha > 1$, the admissible approach region at $\zeta \in \mathbb{S}_n$ is

$$\Gamma(\zeta) = \Gamma_\alpha(\zeta) = \{z \in \mathbb{B}_n : |1 - \langle z, w\rangle| < \frac{\alpha}{2}(1 - |z|^2)\}$$

The admissible maximal function is

$$f^*(\zeta) = f^*_\alpha(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|, \qquad \zeta \in \mathbb{S}_n$$

- Let $0 < p < \infty$ and $f \in H(\mathbb{B}_n)$. Then

$$\|f^*\|_{L^p(\mathbb{S}_n)} \leq C\,\|f\|_{H^p}.$$

The area function

The admissible area function is defined on $\mathbb{S}_n$ by

$$Af(\zeta) = \left(\int_{\Gamma(\zeta)} |Rf(z)|^2 \, (1 - |z|^2)^{1-n} dv(z) \right)^{1/2}$$

The following is a classical result.

Theorem

Let $0 < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then $g \in H^p$ if and only if $Ag \in L^p(\mathbb{S}_n)$. Moreover, if $g(0) = 0$ then

$$\|g\|_{H^p} \asymp \|Ag\|_{L^p(\mathbb{S}_n)}$$

The area function

The admissible area function is defined on $\mathbb{S}_n$ by

$$Af(\zeta) = \left(\int_{\Gamma(\zeta)} |Rf(z)|^2 \, (1 - |z|^2)^{1-n} dv(z) \right)^{1/2}$$

The following is a classical result.

Theorem

Let $0 < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then $g \in H^p$ if and only if $Ag \in L^p(\mathbb{S}_n)$. Moreover, if $g(0) = 0$ then

$$\|g\|_{H^p} \asymp \|Ag\|_{L^p(\mathbb{S}_n)}$$

Carleson measures

A positive measure μ on $\mathbb{B}_n$ is a Carleson measure if

$$\mu(B_\delta(\zeta)) \leq C\delta^n$$

for all $\zeta \in \mathbb{S}_n$ and $\delta > 0$, where

$$B_\delta(\zeta) = \{z \in \mathbb{B}_n : |1 - \langle z, \zeta\rangle| < \delta\}.$$

Theorem [Carleson; Hörmander]

Let $0 < p < \infty$. Then $I_d : H^p(\mathbb{B}_n) \to L^p(\mathbb{B}_n, d\mu)$ is bounded if and only if μ is a Carleson measure.

Bounded Mean Oscillation

The space of analytic functions of bounded mean oscillation $BMOA = BMOA(\mathbb{B}_n)$ consists of those functions $f \in H^1$ with

$$\|f\|_{BMOA} = |f(0)| + \sup \frac{1}{\sigma(Q)} \int_Q |f(\zeta) - f_Q| \, d\sigma(\zeta) < \infty,$$

where

$$f_Q = \frac{1}{\sigma(Q)} \int_Q f \, d\sigma$$

is the mean of f over Q and the supremum is taken over the non-isotropic metric balls

$$Q = Q(\zeta, \delta) = \{\xi \in \mathbb{S}_n : |1 - \langle \zeta, \xi \rangle| < \delta\}$$

for all $\zeta \in \mathbb{S}_n$ and $\delta > 0$.

Bounded Mean Oscillation

The following is a well known result describing *BMOA* in terms of Carleson measures.

Theorem

Let $g \in H(\mathbb{B}_n)$ and consider the measure μ_g defined by

$$d\mu_g(z) = |Rg(z)|^2 (1 - |z|^2)\, dv(z).$$

Then $g \in BMOA$ if and only if μ_g is a Carleson measure. Moreover, if $g(0) = 0$, for all $0 < p < \infty$ one has

$$\|g\|_{BMOA} \asymp \sup_{\|f\|_{H^p}=1} \left(\int_{\mathbb{B}_n} |f(z)|^p \, d\mu_g(z) \right)^{1/2}.$$

J_g bounded in H^p implies g in $BMOA$

Case $0 < p < 2$

We want to show

$$\|g\|_{BMOA} \leq C \, \|J_g\|.$$

Let $f \in H^p$. From the Hardy-Stein inequalities we have

$$\|J_g f\|_{H^p}^p \asymp \int_{\mathbb{B}_n} |J_g f(z)|^{p-2} \, |R(J_g f)(z)|^2 \, (1 - |z|^2) \, dv(z)$$

$$= \int_{\mathbb{B}_n} |J_g f(z)|^{p-2} \, |f(z)|^2 \, d\mu_g(z).$$

Recall that

$$\|g\|_{BMOA}^2 \asymp \sup_{\|f\|_{H^p}=1} \int_{\mathbb{B}_n} |f(z)|^p \, d\mu_g(z)$$

J_g bounded in H^p implies g in $BMOA$

Case $0 < p < 2$

We want to show

$$\|g\|_{BMOA} \le C\,\|J_g\|.$$

Let $f \in H^p$. From the Hardy-Stein inequalities we have

$$\begin{aligned}\|J_g f\|_{H^p}^p &\asymp \int_{\mathbb{B}_n} |J_g f(z)|^{p-2}\,|R(J_g f)(z)|^2\,(1-|z|^2)\,dv(z)\\ &= \int_{\mathbb{B}_n} |J_g f(z)|^{p-2}\,|f(z)|^2\,d\mu_g(z).\end{aligned}$$

Recall that

$$\|g\|_{BMOA}^2 \asymp \sup_{\|f\|_{H^p}=1} \int_{\mathbb{B}_n} |f(z)|^p\,d\mu_g(z)$$

J_g bounded in H^p implies g in $BMOA$

Case $0 < p < 2$

We want to show

$$\|g\|_{BMOA} \le C \, \|J_g\|.$$

Let $f \in H^p$. From the Hardy-Stein inequalities we have

$$\begin{aligned}\|J_g f\|_{H^p}^p &\asymp \int_{\mathbb{B}_n} |J_g f(z)|^{p-2} \, |R(J_g f)(z)|^2 \, (1 - |z|^2) \, dv(z) \\ &= \int_{\mathbb{B}_n} |J_g f(z)|^{p-2} \, |f(z)|^2 \, d\mu_g(z).\end{aligned}$$

Recall that

$$\|g\|_{BMOA}^2 \asymp \sup_{\|f\|_{H^p}=1} \int_{\mathbb{B}_n} |f(z)|^p \, d\mu_g(z)$$

J_g bounded in H^p implies g in $BMOA$

Case $0 < p < 2$

$$\int_{\mathbb{B}_n} |f|^p d\mu_g \leq \left(\int_{\mathbb{B}_n} |J_g f|^p d\mu_g \right)^{\frac{2-p}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{p-2} |f|^2 d\mu_g \right)^{p/2}$$

$$\lesssim \left(\|g\|^2_{BMOA} \cdot \|J_g f\|^p_{H^p} \right)^{\frac{2-p}{2}} \left(\|J_g f\|^p_{H^p} \right)^{p/2}$$

$$= \|g\|^{2-p}_{BMOA} \cdot \|J_g f\|^p_{H^p}$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|^2_{BMOA} \leq C \|g\|^{2-p}_{BMOA} \cdot \|J_g\|^p.$$

J_g bounded in H^p implies g in $BMOA$

Case $0 < p < 2$

$$\int_{\mathbb{B}_n} |f|^p d\mu_g \leq \left(\int_{\mathbb{B}_n} |J_g f|^p d\mu_g \right)^{\frac{2-p}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{p-2} |f|^2 d\mu_g \right)^{p/2}$$

$$\lesssim \left(\|g\|^2_{BMOA} \cdot \|J_g f\|^p_{H^p} \right)^{\frac{2-p}{2}} \left(\|J_g f\|^p_{H^p} \right)^{p/2}$$

$$= \|g\|^{2-p}_{BMOA} \cdot \|J_g f\|^p_{H^p}$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|^2_{BMOA} \leq C \|g\|^{2-p}_{BMOA} \cdot \|J_g\|^p.$$

J_g bounded in H^p implies g in $BMOA$
Case $0 < p < 2$

$$\int_{\mathbb{B}_n} |f|^p d\mu_g \le \left(\int_{\mathbb{B}_n} |J_g f|^p d\mu_g\right)^{\frac{2-p}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{p-2} |f|^2 d\mu_g\right)^{p/2}$$

$$\lesssim \left(\|g\|^2_{BMOA} \cdot \|J_g f\|^p_{H^p}\right)^{\frac{2-p}{2}} \left(\|J_g f\|^p_{H^p}\right)^{p/2}$$

$$= \|g\|^{2-p}_{BMOA} \cdot \|J_g f\|^p_{H^p}$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|^2_{BMOA} \le C \|g\|^{2-p}_{BMOA} \cdot \|J_g\|^p.$$

J_g bounded in H^p implies g in $BMOA$

Case $0 < p < 2$

$$\int_{\mathbb{B}_n} |f|^p d\mu_g \leq \left(\int_{\mathbb{B}_n} |J_g f|^p d\mu_g \right)^{\frac{2-p}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{p-2} |f|^2 d\mu_g \right)^{p/2}$$

$$\lesssim \left(\|g\|^2_{BMOA} \cdot \|J_g f\|^p_{H^p} \right)^{\frac{2-p}{2}} \left(\|J_g f\|^p_{H^p} \right)^{p/2}$$

$$= \|g\|^{2-p}_{BMOA} \cdot \|J_g f\|^p_{H^p}$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|^2_{BMOA} \leq C \|g\|^{2-p}_{BMOA} \cdot \|J_g\|^p.$$

J_g bounded in H^p implies g in $BMOA$

Case $0 < p < 2$

$$\int_{\mathbb{B}_n} |f|^p d\mu_g \le \left(\int_{\mathbb{B}_n} |J_g f|^p d\mu_g \right)^{\frac{2-p}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{p-2} |f|^2 d\mu_g \right)^{p/2}$$

$$\lesssim \left(\|g\|^2_{BMOA} \cdot \|J_g f\|^p_{H^p} \right)^{\frac{2-p}{2}} \left(\|J_g f\|^p_{H^p} \right)^{p/2}$$

$$= \|g\|^{2-p}_{BMOA} \cdot \|J_g f\|^p_{H^p}$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|^2_{BMOA} \le C \|g\|^{2-p}_{BMOA} \cdot \|J_g\|^p.$$

The case $0 < q < p < \infty$

Let $g \in H(\mathbb{B}_n)$ and $0 < q < p < \infty$. Then $J_g : H^p(\mathbb{B}_n) \to H^q(\mathbb{B}_n)$ is bounded if and only if $g \in H^r(\mathbb{B}_n)$ with $r = pq/(p-q)$. Moreover,

$$\|J_g\|_{H^p \to H^q} \asymp \|g\|_{H^r}.$$

Sufficiency:

$$\|J_g f\|^q_{H^q} \asymp \int_{\mathbb{S}_n} \left(\int_{\Gamma(\zeta)} |f|^2 |Rg|^2 (1-|z|^2)^{1-n} dv(z) \right)^{q/2} d\sigma(\zeta)$$

$$\leq \int_{\mathbb{S}_n} |f^*(\zeta)|^q \left(\int_{\Gamma(\zeta)} |Rg|^2 (1-|z|^2)^{1-n} dv(z) \right)^{q/2} d\sigma(\zeta)$$

$$\leq \|f^*\|^q_{L^p(\mathbb{S}_n)} \cdot \|Ag\|^q_{L^r(\mathbb{S}_n)}$$

Luecking's theorem

Theorem

Let $0 < s < p < \infty$ and let μ be a positive Borel measure on $\mathbb{B}_n$. Then the identity $I_d : H^p \to L^s(\mu)$ is bounded, if and only if, the function defined on $\mathbb{S}_n$ by

$$\widetilde{\mu}(\zeta) = \int_{\Gamma(\zeta)} (1 - |z|^2)^{-n} d\mu(z)$$

belongs to $L^{p/(p-s)}(\mathbb{S}_n)$. Moreover, one has

$$\|I_d\|_{H^p \to L^s(\mu)} \asymp \|\widetilde{\mu}\|^{1/s}_{L^{p/(p-s)}(\mathbb{S}_n)}.$$

An immediate consequence of that and the area theorem is

Corollary

Let $0 < s < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then

$$\int_{\mathbb{B}_n} |f(z)|^s d\mu_g(z) \le C\|f\|_{H^p}^s$$

if and only if $g \in H^{\frac{2p}{p-s}}$. Moreover, $\|I_d\|_{H^p \to L^s(\mu_g)} \asymp \|g\|_{H^{\frac{2p}{p-s}}}^{2/s}$.

Take s with $r = 2p/(p-s)$, that is,

$$s = p - 2(p-q)/q.$$

If $0 < q < 2$ then $0 < s < 2$ and

$$\int_{\mathbb{B}_n} |f|^s d\mu_g \le \left(\int_{\mathbb{B}_n} |J_g f|^{\frac{s(2-q)}{2-s}} d\mu_g \right)^{\frac{2-s}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{q-2} |f|^2 d\mu_g \right)^{\frac{s}{2}}$$

Corollary

Let $0 < s < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then

$$\int_{\mathbb{B}_n} |f(z)|^s d\mu_g(z) \le C\|f\|_{H^p}^s$$

if and only if $g \in H^{\frac{2p}{p-s}}$. Moreover, $\|I_d\|_{H^p \to L^s(\mu_g)} \asymp \|g\|_{H^{\frac{2p}{p-s}}}^{2/s}$.

Take s with $r = 2p/(p-s)$, that is,

$$s = p - 2(p-q)/q.$$

If $0 < q < 2$ then $0 < s < 2$ and

$$\int_{\mathbb{B}_n} |f|^s d\mu_g \le \left(\int_{\mathbb{B}_n} |J_g f|^{\frac{s(2-q)}{2-s}} d\mu_g\right)^{\frac{2-s}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{q-2} |f|^2 d\mu_g\right)^{\frac{s}{2}}$$

Case $0 < q < p < \infty$: necessity
The case $r > 2$

Corollary

Let $0 < s < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then

$$\int_{\mathbb{B}_n} |f(z)|^s d\mu_g(z) \leq C\|f\|^s_{H^p}$$

if and only if $g \in H^{\frac{2p}{p-s}}$. Moreover, $\|I_d\|_{H^p \to L^s(\mu_g)} \asymp \|g\|^{2/s}_{H^{\frac{2p}{p-s}}}$.

Take s with $r = 2p/(p-s)$, that is,

$$s = p - 2(p-q)/q.$$

If $0 < q < 2$ then $0 < s < 2$ and

$$\int_{\mathbb{B}_n} |f|^s d\mu_g \leq \left(\int_{\mathbb{B}_n} |J_g f|^{\frac{s(2-q)}{2-s}} d\mu_g\right)^{\frac{2-s}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{q-2} |f|^2 d\mu_g\right)^{\frac{s}{2}}$$

Case $0 < q < p < \infty$: necessity for $r > 2$

$$\int_{\mathbb{B}_n} |f|^s d\mu_g \leq \left(\int_{\mathbb{B}_n} |J_g f|^{\frac{s(2-q)}{2-s}} \, d\mu_g \right)^{\frac{2-s}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{q-2} |f|^2 d\mu_g \right)^{\frac{s}{2}}$$

$$\lesssim \left(\|J_g f\|_{H^q}^{\frac{s(2-q)}{2-s}} \cdot \|g\|_{H^r}^2 \right)^{\frac{2-s}{2}} \cdot \left(\|J_g f\|_{H^q}^q \right)^{s/2}$$

$$= \|g\|_{H^r}^{2-s} \cdot \|J_g f\|_{H^q}^s$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|_{H^r}^2 \leq C \|g\|_{H^r}^{2-s} \cdot \|J_g\|_{H^p \to H^q}^s.$$

Case $0 < q < p < \infty$: necessity for $r > 2$

$$\int_{\mathbb{B}_n} |f|^s d\mu_g \le \left(\int_{\mathbb{B}_n} |J_g f|^{\frac{s(2-q)}{2-s}} \, d\mu_g \right)^{\frac{2-s}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{q-2} |f|^2 d\mu_g \right)^{\frac{s}{2}}$$

$$\lesssim \left(\|J_g f\|_{H^q}^{\frac{s(2-q)}{2-s}} \cdot \|g\|_{H^r}^2 \right)^{\frac{2-s}{2}} \cdot \left(\|J_g f\|_{H^q}^q \right)^{s/2}$$

$$= \|g\|_{H^r}^{2-s} \cdot \|J_g f\|_{H^q}^s$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|_{H^r}^2 \le C \|g\|_{H^r}^{2-s} \cdot \|J_g\|_{H^p \to H^q}^s.$$

Case $0 < q < p < \infty$: necessity for $r > 2$

$$\int_{\mathbb{B}_n} |f|^s d\mu_g \le \left(\int_{\mathbb{B}_n} |J_g f|^{\frac{s(2-q)}{2-s}} \, d\mu_g \right)^{\frac{2-s}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{q-2} |f|^2 d\mu_g \right)^{\frac{s}{2}}$$

$$\lesssim \left(\|J_g f\|_{H^q}^{\frac{s(2-q)}{2-s}} \cdot \|g\|_{H^r}^2 \right)^{\frac{2-s}{2}} \cdot \left(\|J_g f\|_{H^q}^q \right)^{s/2}$$

$$= \|g\|_{H^r}^{2-s} \cdot \|J_g f\|_{H^q}^s$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|_{H^r}^2 \le C \|g\|_{H^r}^{2-s} \cdot \|J_g\|_{H^p \to H^q}^s.$$

Case $0 < q < p < \infty$: necessity for $r > 2$

$$\int_{\mathbb{B}_n} |f|^s d\mu_g \le \left(\int_{\mathbb{B}_n} |J_g f|^{\frac{s(2-q)}{2-s}} \, d\mu_g \right)^{\frac{2-s}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{q-2} |f|^2 d\mu_g \right)^{\frac{s}{2}}$$

$$\lesssim \left(\|J_g f\|_{H^q}^{\frac{s(2-q)}{2-s}} \cdot \|g\|_{H^r}^2 \right)^{\frac{2-s}{2}} \cdot \left(\|J_g f\|_{H^q}^q \right)^{s/2}$$

$$= \|g\|_{H^r}^{2-s} \cdot \|J_g f\|_{H^q}^s$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|_{H^r}^2 \le C \|g\|_{H^r}^{2-s} \cdot \|J_g\|_{H^p \to H^q}^s.$$

Case $0 < q < p < \infty$: necessity for $r > 2$

Hardy spaces integration operators

Jordi Pau

$$\int_{\mathbb{B}_n} |f|^s d\mu_g \leq \left(\int_{\mathbb{B}_n} |J_g f|^{\frac{s(2-q)}{2-s}} d\mu_g \right)^{\frac{2-s}{2}} \left(\int_{\mathbb{B}_n} |J_g f|^{q-2} |f|^2 d\mu_g \right)^{\frac{s}{2}}$$
$$\lesssim \left(\|J_g f\|_{H^q}^{\frac{s(2-q)}{2-s}} \cdot \|g\|_{H^r}^2 \right)^{\frac{2-s}{2}} \cdot \left(\|J_g f\|_{H^q}^q \right)^{s/2}$$
$$= \|g\|_{H^r}^{2-s} \cdot \|J_g f\|_{H^q}^s$$

Taking the supremum over all $f \in H^p$ with $\|f\|_{H^p} = 1$,

$$\|g\|_{H^r}^2 \leq C \|g\|_{H^r}^{2-s} \cdot \|J_g\|_{H^p \to H^q}^s.$$

Necessity for $r \leq 2$

Corollary

Let $0 < s < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then

$$\int_{\mathbb{B}_n} |f(z)|^s d\mu_g(z) \leq C\|f\|_{H^p}^s$$

if and only if $g \in H^{\frac{2p}{p-s}}$. Moreover, $\|I_d\|_{H^p \to L^s(\mu_g)} \asymp \|g\|_{H^{\frac{2p}{p-s}}}^{2/s}$.

Assume that g has no zeros. Then

$$g \in H^r \quad \Leftrightarrow \quad g^t \in H^{r/t}$$

Take $t > 0$ so that $r/t > 2$ and let s with $r/t = 2p/(p-s)$. Then

$$g \in H^r \Leftrightarrow \int_{\mathbb{B}_n} |f|^s \, |g|^{2t-2} d\mu_g \leq C\|f\|_{H^p}^s$$

Necessity for $r \leq 2$

Corollary

Let $0 < s < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then

$$\int_{\mathbb{B}_n} |f(z)|^s d\mu_g(z) \leq C\|f\|_{H^p}^s$$

if and only if $g \in H^{\frac{2p}{p-s}}$. Moreover, $\|I_d\|_{H^p \to L^s(\mu_g)} \asymp \|g\|_{H^{\frac{2p}{p-s}}}^{2/s}$.

Assume that g has no zeros. Then

$$g \in H^r \quad \Leftrightarrow \quad g^t \in H^{r/t}$$

Take $t > 0$ so that $r/t > 2$ and let s with $r/t = 2p/(p-s)$. Then

$$g \in H^r \Leftrightarrow \int_{\mathbb{B}_n} |f|^s \, |g|^{2t-2} d\mu_g \leq C\|f\|_{H^p}^s$$

Necessity for $r \le 2$

Corollary

Let $0 < s < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then

$$\int_{\mathbb{B}_n} |f(z)|^s d\mu_g(z) \le C\|f\|_{H^p}^s$$

if and only if $g \in H^{\frac{2p}{p-s}}$. Moreover, $\|I_d\|_{H^p \to L^s(\mu_g)} \asymp \|g\|^{2/s}_{H^{\frac{2p}{p-s}}}$.

Assume that g has no zeros. Then

$$g \in H^r \quad \Leftrightarrow \quad g^t \in H^{r/t}$$

Take $t > 0$ so that $r/t > 2$ and let s with $r/t = 2p/(p-s)$. Then

$$g \in H^r \Leftrightarrow \int_{\mathbb{B}_n} |f|^s \, |g|^{2t-2} d\mu_g \le C\|f\|_{H^p}^s$$

Necessity for $r \leq 2$

Theorem

Let $0 < s < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then

$$\int_{\mathbb{B}_n} |f(z)|^s |g(z)|^{2t-2} d\mu_g(z) \leq C \|f\|_{H^p}^s$$

if and only if $g \in H^{\frac{2pt}{p-s}}$.

Moreover, if $\widehat{\mu}_g$ is the measure defined by

$$d\widehat{\mu}_g(z) = |g(z)|^{2t-2}\, d\mu_g(z),$$

then

$$\| I_d \|_{H^p \to L^s(\widehat{\mu}_g)} \asymp \|g\|^{2t/s}_{H^{\frac{2pt}{p-s}}}.$$

General Area function description of Hardy spaces

Theorem

Let $g \in H(\mathbb{B}_n)$ and $0 < p, t < \infty$. Then $g \in H^{pt}$ if and only if

$$\int_{\mathbb{S}_n} \left(\int_{\Gamma(\zeta)} |g(z)|^{2t-2} \, |Rg(z)|^2 (1 - |z|^2)^{1-n} dv(z) \right)^{p/2} d\sigma(\zeta) < \infty.$$