Trace ideal criteria for Toeplitz operators on weighted Dirichlet spaces

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Hilbert Functions spaces
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Outline of the talk

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- Toeplitz operators on classical Dirichlet spaces;

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- Toeplitz operators on weighted Dirichlet spaces (more general setting).

- Let H be a separable Hilbert space and $T : H \to H$ bounded, let λ_n for any non-negative integer n, the n:th *singular value* of T.

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- For $0 < p < \infty$, the *Schatten p-class*

$$\mathcal{S}_p(H) = \left\{ T : \|\{\lambda_n\}_{n=0}^{\infty}\|_{\ell^p} = \left(\sum_{n=0}^{\infty} |\lambda_n|^p \right)^{1/p} < \infty. \right\}.$$

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- H_0 is the Hardy space H^2,
- If $\alpha < 0$, $H_\alpha = A^2_{-1-\alpha}$ (Bergman space)

- If $\{e_n\}$ is an o. b. of H_α, space of analytic functions in $\mathbb{D}$ with rep. kernel K_z^α, then

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Problem

Which are those positive Borel measures such that $Q_\mu \in S_p(H_\alpha)$?

Luecking's contribution

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A decomposition of $\mathbb{D}$

Let Υ denote the family of all dyadic arcs of $\mathbb{T}$,

$$I = I_{n,k} = \left\{ e^{i\theta} : \frac{2\pi k}{2^n} \leq \theta < \frac{2\pi(k+1)}{2^n} \right\},$$

where $k = 0, 1, 2, \dots, 2^n - 1$ and $n \in \mathbb{N} \cup \{0\}$. For each $I \subset \mathbb{T}$, set

$$R(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I,\ 1 - \frac{|I|}{2\pi} \leq r < 1 - \frac{|I|}{4\pi} \right\}.$$

Then $\{R(I) : I \in \Upsilon\}$ consists of pairwise disjoint sets whose union covers $\mathbb{D}$. For $I_j \in \Upsilon \setminus \{I_{0,0}\}$, we will write z_j for the unique point in $\mathbb{D}$ such that $z_j = (1 - |I_j|/2\pi) a_j$, where $a_j \in \mathbb{T}$ is the midpoint of I_j

Theorem. (Luecking 1987)

Let $0 < p < \infty$ and $-\infty < \alpha < 1$ such that $p\alpha < 1$, let μ be a positive Borel measure on $\mathbb{D}$. If

$$\sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{(1 - |z_j|)^{(1-\alpha)}} \right)^p < \infty, \tag{1}$$

then $Q_\mu \in \mathcal{S}_p(H_\alpha)$, and there exists a constant $C > 0$ such that

$$|Q_\mu|_p^p \leq C \sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{(1 - |z_j|)^{(1-\alpha)}} \right)^p .$$

Conversely, if μ is a positive Borel measure on $\mathbb{D}$ and $Q_\mu \in \mathcal{S}_p(H_\alpha)$, then (1) is satisfied.

Question

What about the case $p\alpha \geq 1$?

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- The hypotheses $p\alpha < 1$ is strongly used.

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- Calderón (60's)

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- Pommerenke (70's). $T_g : H^2 \to H^2$ is bounded if and only if $g \in BMOA$.
- Coifman and Meyer (1978).
- Aleman and (several authors) have obtained deeper results on T_g.

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Theorem

Let $g \in \mathcal{H}(\mathbb{D})$. We have the following:

(a) Let $1 < p$ and $-\infty < \alpha < 1$ with $p\alpha < 2$. Then $T_g \in S_p(H_\alpha)$ if and only if g belongs to B_p.

(b) If $0 < p \leq 1$ and $-\infty < \alpha < 1$, then $T_g \in S_p(H_\alpha)$ if and only if g is constant.

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- If $p\alpha \geq 2$, $B_p \not\subset H_\alpha$

- For each $\alpha < 2$ and $g \in \mathcal{H}(\mathbb{D})$, $T_g^\star T_g = Q_{\mu_g}$, where $\mu_g = |g'(z)|^2(1-|z|^2)^{1-\alpha}$.
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- If $p\alpha \geq 2$, $B_p \not\subset H_\alpha$
- Luecking's theorem does not remain for $p\alpha \geq 1$!!!!

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- $$J_z(\zeta) = \frac{\partial}{\partial \bar{z}} K_z(\zeta) = \sum_n e_n(\zeta)\, \overline{e_n'(z)}.$$
- $$j_z(\zeta) = \frac{J_z(\zeta)}{||J_z||_H}.$$

Proposition

Let $T : H_\alpha \to H$ be a compact operator, where H is any separable Hilbert space.

(i) For $p \geq 2$,

$$\int_{\mathbb{D}} \| T j_z^\alpha \|_H^p \frac{dA(z)}{(1-|z|^2)^2} \leq \frac{1}{2-\alpha} \| T \|^p_{\mathcal{S}_p(H_\alpha)}.$$

(ii) For $0 < p \leq 2$,

$$\| T \|^p_{\mathcal{S}_p(H_\alpha)} \leq \| T \|^p + (2-\alpha) \int_{\mathbb{D}} \| T j_z^\alpha \|_H^p \frac{dA(z)}{(1-|z|^2)^2}.$$

Proposition. Pau-P (2012)

Let $0 < \alpha < 1$ and $g \in \mathcal{H}(\mathbb{D})$, then

(i) If $p \geq 2$ then

$$\int_{\mathbb{D}} \left((1-|w|^2)^{1-\alpha} \int_{\mathbb{D}} \frac{|g'(z)|^2 (1-|z|^2)^{1-\alpha}\, dA(z)}{|1-\bar{w}z|^{4-2\alpha}} \right)^{p/2} (1-|w|^2)^{p-2}\, dA(w)$$
$$\lesssim \|T_g\|^p_{\mathcal{S}_p}.$$

(ii) For $0 < p \leq 2$,

$$\|T_g\|^p_{\mathcal{S}_p} \lesssim |g(0)|^p +$$
$$\int_{\mathbb{D}} \left((1-|w|^2)^{1-\alpha} \int_{\mathbb{D}} \frac{|g'(z)|^2 (1-|z|^2)^{1-\alpha}\, dA(z)}{|1-\bar{w}z|^{4-2\alpha}} \right)^{p/2} (1-|w|^2)^{p-2}\, dA(w)$$

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Proposition. Pau-P (2012)

Let $1 < p < \infty$ and $\alpha \leq 1$. Then

1. $Y_\alpha^p \subset H_\alpha \cap B_p$.

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2. If $p < q$, then $Y^p_\alpha \subset Y^q_\alpha$.

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3. If $\gamma < \alpha \le 1$, then $Y^p_\alpha \subset Y^p_\gamma$.
4. Let $\alpha < 1$ and $p\alpha < 2$ then $Y^p_\alpha = B_p$.

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2. If $p < q$, then $Y_\alpha^p \subset Y_\alpha^q$.
3. If $\gamma < \alpha \leq 1$, then $Y_\alpha^p \subset Y_\gamma^p$.
4. Let $\alpha < 1$ and $p\alpha < 2$ then $Y_\alpha^p = B_p$.
5. The previous assertion is not longer true for $p\alpha \geq 2$.

Theorem. Pau-P (2012)

Let $0 < \alpha < 1$, $g \in H(\mathbb{D})$ and $p > 1$ with $p\alpha < 4$. Then $T_g \in S_p(H_\alpha)$ if and only if g belongs to Y_α^p.

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- The proof is rather technical.

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Let $0 < \alpha < 1$, $g \in H(\mathbb{D})$ and $p > 1$ with $p\alpha < 4$. Then $T_g \in S_p(H_\alpha)$ if and only if g belongs to Y^p_α.

- The proof is rather technical.
- Further results on several operators.

$$Q_\mu f(z) = \int_{\mathbb{D}} f(\zeta)\, \overline{K_z^\alpha(\zeta)}\, d\mu(\zeta).$$

Theorem. Pau-P (2012)

Let μ be a finite positive Borel measure on $\mathbb{D}$, $\alpha < 1$, and let $p > 0$ with $1 < p(3-\alpha)$ and $p\alpha < 2$. Then the Toeplitz operator Q_μ belongs to $\mathcal{S}_p(H_\alpha)$ if and only if

$$\int_{\mathbb{D}} \left((1-|w|^2)^{1-\alpha} \int_{\mathbb{D}} \frac{d\mu(z)}{|1-\bar{w}z|^{4-2\alpha}} \right)^p (1-|w|^2)^{2p-2} dA(w) < \infty.$$

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- $\omega : \mathbb{D} \to (0, \infty)$, integrable over $\mathbb{D}$, is called a *weight*. It is *radial* if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$.
- The *weighted Bergman space* A^2_ω consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

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- L-P formula for radial weights,

$$\|f\|^2_{A^2_\omega} = 4\|f'\|^2_{A^2_{\omega^\star}} + \omega(\mathbb{D})|f(0)|^2.$$

Definition

A radial continuous weight ω is called *rapidly increasing*, $\omega \in \mathcal{I}$, if

$$\lim_{r \to 1^-} \frac{\omega^\star(r)}{\omega(r)(1-r)^2} = \infty.$$

A radial continuous weight ω is called *regular*, $\omega \in \mathcal{R}$, if

$$C_1 \leq \frac{\omega^\star(r)}{\omega(r)(1-r)^2} \leq C_2, \quad 0 < r < 1.$$

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Problem

Which are those analytic functions $g \in \mathcal{H}(\mathbb{D})$ such that $T_g \in S_p(A^2_\omega)$?

- For $\alpha \in \mathbb{R}$ and $\omega \in \mathcal{I} \cup \mathcal{R}$, $H_\alpha(\omega^\star)$ consists of those $f(z) = \sum_{n=0}^\infty a_n z^n$ such that

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- $H_0(\omega^\star) = A^2_\omega$ (L-P formula).
- There exist reproducing kernels $K_a^\alpha \in H_\alpha(\omega^\star)$ with $f(a) = \langle f, K_a^\alpha \rangle_{H_\alpha(\omega^\star)}$.

- Toeplitz operator $Q_\mu(f)(w) = \int_{\mathbb{D}} f(z) K^\alpha(w,z)\, d\mu(z), \quad f \in H_\alpha(\omega^\star),$

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Problem

Which are those positive Borel measures such that $Q_\mu \in S_p(H_\alpha(\omega^\star))$?

P-Rättyä. (2011-2013)

Let $0 < p < \infty$ and $-\infty < \alpha < 1$ such that $p\alpha < 1$. Let $\omega \in \mathcal{I} \cup \mathcal{R}$, and let μ be a positive Borel measure on $\mathbb{D}$. If

$$\sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{(1-|z_j|)^{-\alpha}\omega^\star(z_j)} \right)^p < \infty, \tag{2}$$

then $Q_\mu \in \mathcal{S}_p(H_\alpha(\omega^\star))$, and there exists a constant $C > 0$ such that

$$|Q_\mu|_p^p \leq C \sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{(1-|z_j|)^{-\alpha}\omega^\star(z_j)} \right)^p .$$

Conversely, if μ is a positive Borel measure on $\mathbb{D}$ and $Q_\mu \in \mathcal{S}_p(H_\alpha(\omega^\star))$, then (2) is satisfied.

P-Rättyä. (2013)

Let $0 < p < \infty$, $\omega \in \mathcal{I} \cup \mathcal{R}$, $-\infty < \alpha < 1$ and $N \in \mathbb{N}$ such that $(1-|z|)^{Np-2-\frac{p\alpha}{2}} \left(\omega^\star(z)\right)^{\frac{p}{2}}$ is a regular weight. Then

$$\int_{\mathbb{D}} \left| \frac{\partial^N K^\alpha(z,a)}{\partial^N z} \right|^p (1-|z|)^{Np-2-\frac{p\alpha}{2}} \left(\omega^\star(z)\right)^{\frac{p}{2}} \, dA(z) \asymp \left((1-|a|)^{-\alpha} \omega^\star(a)\right)^{-\frac{p}{2}}$$

for all $a \in \mathbb{D}$.

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- Smooth polynomials (Cesáro-type). Hadamard products.

Key tools

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- If $-\infty < \alpha < 1$ and $\omega \in \mathcal{I} \cup \mathcal{R}$, then

$$\|K_a^\alpha\|^2_{H_\alpha(\omega^\star)} \asymp \frac{1}{(1-|a|)^{-\alpha}\omega^\star(a)} \asymp \frac{(1-|a|)^\alpha}{\omega(S((a))}, \quad |a| \geq \frac{1}{2}.$$