# Hankel Forms and Embedding Theorems in Weighted Dirichlet Spaces

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# Classical Hankel forms

A holomorphic symbol $g\colon \mathbb{D} \to \mathbb{C}$, $g(0) = 0$, induces a Hankel form on $H^2$ by

$$(f,h)_g = \lim_{r\to 1^-} \int_{\mathbb{T}} f(z)\overline{h(\bar{z})g(rz)}\frac{ds(z)}{2\pi}, \quad f,h \in H^2.$$

The form induces a corresponding Hankel operator $\Gamma_g$,

$$\langle \Gamma_g f, h\rangle_{H^2} = (f,h)_g.$$

Writing $\Gamma_g \sim (A_{ij})$ as a matrix, $A_{ij}$ depends only on $i+j$,

$$A_{ij} = \overline{\hat{g}(i+j)}, \quad g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n, \quad z \in \mathbb{D}.$$

# Classical Hankel forms

## Question

Which symbols $g$ induce a bounded Hankel operator $\Gamma_g : H^2 \to H^2$?
Equivalently, when is there $0 < C < \infty$ such that

$$|(f, h)_g| \leq C\|f\|_{H^2}\|h\|_{H^2},$$

so that $(f, h) \to (f, h)_g$ is a bounded Hankel form?

# Classical Hankel forms – boundedness

- Carleson: A measure $\mu$ on $\mathbb{D}$ is a Carleson measure for $H^2$ iff there is $C < \infty$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq C \|f\|_{H^2}^2, \quad f \in H^2,$$

  i.e. the embedding of $H^2$ into $L^2(\mu)$ is bounded.
- $g \in BMOA$ if and only if $d\mu_g(z) = |g'(z)|^2(1 - |z|^2) \, dA(z)$ is a *Carleson measure* for $H^2$, where $dA(z)$ is area measure on $\mathbb{D}$.

**Theorem**

$$\sup_{\|f\|_2 = \|h\|_2 = 1} |(f, h)_g|^2 \sim \|g\|_{\mathrm{BMOA}}^2 \sim \sup_{\|f\|_2 = 1} \int_{\mathbb{D}} |f(z) g'(z)|^2 (1 - |z|^2) \, dA(z).$$

# Dirichlet spaces

For $p > 1$ and $\beta > -1$, the weighted Dirichlet space $D^{p,\beta}$ consists of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{p,\beta}^{p} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{\beta}\, dA(z) < \infty.$$

# Classical Hankel forms revisited

Recall Littlewood-Paley identity:

$$\|f\|_{H^2} = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|}\, dA(z).$$

Therefore $H^2 = D^{2,1}$, $\|f\|_{H^2} \sim \|f\|_{2,1}$. We may equip $H^2$ with alternative scalar product

$$\langle f, h\rangle_{2,1} = f(0)\overline{h(0)} + \int_{\mathbb{D}} f'(z)\overline{h'(z)}(1-|z|^2)\, dA(z).$$

## Theorem (Restated)

$$\sup_{\|f\|_{2,1}=\|h\|_{2,1}=1} \left|\lim_{r\to 1}\int_{\mathbb{D}_r} (fh)'(z)\overline{g'(z)}(1-|z|^2)\, dA(z)\right|^2$$
$$\sim \sup_{\|f\|_{2,1}=1}\int_{\mathbb{D}} |f(z)g'(z)|^2(1-|z|^2)\, dA(z).$$

# Hankel forms on Dirichlet spaces

## Question

Is the boundedness of a Hankel form on the Dirichlet space $D^{2,\beta}$, $0 \leq \beta < 1$, also equivalent to a corresponding Carleson embedding condition?

- Very tough question! Carleson measures for $D^{2,\beta}$ characterized in terms of capacities and difficult to work with.

## Theorem (Arcozzi, Rochberg, Sawyer, Wick '10)

$$\sup_{\|f\|_{2,0}=\|h\|_{2,0}=1} \left| \lim_{r\to 1} \int_{\mathbb{D}_r} (fh)'(z)\overline{g'(z)}\, dA(z) \right|^2 \sim \sup_{\|f\|_{2,0}=1} \int_{\mathbb{D}} |f(z)g'(z)|^2\, dA(z).$$

# Hankel-type forms on Dirichlet spaces

- Consider alternative Hankel-type form

$$(f, h) \to \lim_{r \to 1} \int_{\mathbb{D}_r} f(z)h'(z)\overline{g'(z)}(1 - |z|^2)^\beta \, dA(z), \quad f, h \in D^{2,\beta},$$

in which $(fh)' = f'h + fh'$ is replaced with $fh'$, taking only one "half" of the original Hankel form.

# Hankel-type forms on Dirichlet spaces

- Half-forms considerably easier to analyze!
- Rochberg and Wu '93 proved

$$\sup_{\|f\|_{2,0}=\|h\|_{2,0}=1} \left| \lim_{r\to 1} \int_{\mathbb{D}_r} f(z)h'(z)\overline{g'(z)}\,dA(z) \right|^2 \sim \sup_{\|f\|_{2,0}=1} \int_{\mathbb{D}} |f(z)g'(z)|^2\,dA(z).$$

- The two types of Hankel forms therefore are bounded simultaneously in the $D^{2,0}$.
- For the scalar-valued Hardy space-case it is straightforward to check that (Bourgain's lemma)

$$\sup_{\|f\|_{2,1}=\|h\|_{2,1}=2} \left| \lim_{r\to 1} \int_{\mathbb{D}_r} f(z)h'(z)\overline{g'(z)}(1-|z|^2)\,dA(z) \right|^2 \sim \sup_{\|f\|_{2,1}=1} \int_{\mathbb{D}} |f(z)g'(z)|^2(1-|z|^2)\,dA(z).$$

# Hankel-type forms on Dirichlet spaces

- Let $1 < p, q < \infty$, $\alpha \geq 0$, and $\beta, \gamma > -1$ satisfy the duality relations $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{\beta}{p} + \frac{\gamma}{q} = \alpha$, so that

$$(D^{p,\beta})^* \simeq D^{q,\gamma} \text{ under the } D^{2,\alpha}\text{-pairing.}$$

- Consider boundedness on $D^{p,\beta} \times D^{q,\gamma}$ of form

$$(f,h) \to \lim_{r \to 1} \int_{\mathbb{D}_r} f(z)h'(z)\overline{g'(z)}(1-|z|^2)^\alpha \, dA(z), \; f \in D^{p,\beta}, h \in D^{q,\gamma}.$$

- Corresponds to a (small) Hankel operator with matrix $(A_{ij})$ with

$$A_{ij} \sim i^{\frac{\alpha-1}{2}} j^{\frac{\alpha+1}{2}} (i+j)^{-\alpha} \overline{\hat{g}(i+j)}.$$

# Hankel-type forms on Dirichlet spaces – Scalar case

In Aleman and Perfekt '12 we show that boundedness is equivalent to the Carleson embedding condition for general parameters in the *vector-valued* case. The scalar version of the result reads

## Theorem (Scalar case)

*Let* $1 < p, q < \infty$, $\alpha \geq 0$, *and* $\beta, \gamma > -1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\frac{\beta}{p} + \frac{\gamma}{q} = \alpha$, *and let* $g : \mathbb{D} \to \mathbb{C}$ *be a holomorphic function. Then*

$$\sup_{\|f\|_{p,\beta} = \|h\|_{q,\gamma} = 1} \left| \lim_{r \to 1} \int_{\mathbb{D}_r} f(z) h'(z) \overline{g'(z)} (1 - |z|^2)^{\alpha} \, dA(z) \right|$$
$$\sim \sup_{\|f\|_{p,\beta} = 1} \left( \int_{\mathbb{D}} |\overline{g'(z)} f(z)|^p (1 - |z|^2)^{\beta} \, dA(z) \right)^{1/p}.$$

# Main theorem – abridged

**Theorem (Aleman and Perfekt '12, abridged version)**

*For a Hilbert space $\mathcal{H}$ and $\beta \geq 0$, let $g : \mathbb{D} \to B(\mathcal{H})$ be a holomorphic function. Then*

$$\sup_{\|f\|_{D^{2,\beta}(\mathcal{H})}, \|h\|_{D^{2,\beta}(\mathcal{H})} \leq 1} \left| \lim_{r \to 1} \int_{\mathbb{D}_r} \langle g'(z)^* f(z), h'(\bar{z}) \rangle_{\mathcal{H}} (1 - |z|^2)^\beta \, dA(z) \right|$$

$$\sim \sup_{\|f\|_{D^{2,\beta}(\mathcal{H})} \leq 1} \left( \int_{\mathbb{D}} \|g'(z)^* f(z)\|_{\mathcal{H}}^2 (1 - |z|^2)^\beta \, dA(z) \right)^{1/2}.$$

- When $\dim \mathcal{H} = \infty$, $p = 2$ and $\beta \leq 1$ there are rank-one operator-valued holomorphic functions $g : \mathbb{D} \to B(\mathcal{H})$ such that

$$\sup_{\|f\|_{D^{2,\beta}(\mathcal{H})} \leq 1} \left( \int_{\mathbb{D}} \|g'(z)^* f(z)\|_{\mathcal{H}}^2 (1 - |z|^2)^\beta \, dA(z) \right)^{1/2} = \infty,$$

  but

$$\sup_{\|f\|_{D^{2,\beta}(\mathcal{H})} \leq 1} \left( \int_{\mathbb{D}} \|g'(z) f(z)\|_{\mathcal{H}}^2 (1 - |z|^2)^\beta \, dA(z) \right)^{1/2} < \infty.$$

- In the Bergman space-case $\beta > p - 1$, the Carleson embedding condition simply means that $g$ is in the Bloch space, i.e.

$$\sup_{z \in \mathbb{D}} \|g'(z)\|_{B(\mathcal{H})} (1 - |z|^2) < \infty.$$

  In this case $g'(z)^*$ can be replaced with $g'(z)$.