Krein - de Branges theory in spectral analysis

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Krein's systems

Symplectic structure on $\mathbb{R}^2$: $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\{x, y\} = (\Omega x, y)$.

Consider a 2×2 differential system with a spectral parameter z:

$$\Omega \dot{X} = zH(t)X - Q(t)X, \quad t_- < t < t_+$$

where $X(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$. We assume the (real-valued) coefficients to satisfy

$$H,\ Q \in L^1_{loc}((t_-, t_+) \to \mathbb{R}^{2\times 2}).$$

By definition, a solution $X = X_z(t)$ is a $C^2((t_-, t_+))$-function satisfying the equation.

Theorem

Every IVP has a unique solution on (t_-, t_+). For each fixed t, this solution presents an entire function $u_z(t) + iv_z(t)$ of z of exponential type.

Self-adjoint systems

$$(*)\ \ \Omega\dot{X} = zH(t)X - Q(t)X,\ \ t_- < t < t_+.$$

We may further assume that $H(t), Q(t)$ are real symmetric locally summable matrix-valued functions and that $H(t) \geq 0$. The Hilbert space $L^2(H)$ consists of (equivalence classes) of vector-functions with

$$||f||_H^2 = \int_{t_-}^{t_+} \{Hf, f\} dt < \infty.$$

The system $(*)$ is an eigenvalue equation $DX = zX$ for the (formal) differential operator

$$D = H^{-1}\left[\Omega\frac{d}{dt} + Q\right].$$

Schrödinger equations:

$$-\ddot{u} = zu - qu.$$

Put $v = -\dot{u}$ and $X = (u, v)^T$ to obtain

$$\Omega\dot{X} = z\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}X - \begin{pmatrix}q & 0\\ 0 & -1\end{pmatrix}X.$$

Dirac systems:

$H \equiv I$. The general form is

$$\Omega\dot{X} = z\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}X - \begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}X,\ q_{12} = q_{21}.$$

The "standard form": $Q = \begin{pmatrix}-q_2 & -q_1\\ -q_1 & q_2\end{pmatrix}$. In this case $f = q_1 + iq_2$ is the potential function.

Krein's Canonical Systems

Canonical Systems are self-adjoint systems with $Q \equiv 0$:

$$\Omega \dot{X} = zH(t)X.$$

A general self-adjoint system can be reduced to canonical form:

To reduce

$$\Omega \dot{X} = zH(t)X - Q(t)X, \quad (*)$$

solve $\Omega \dot{V} = -QV$ and make a substitution $X = VY$. Then $(*)$ becomes

$$\Omega \dot{Y} = z\left[V^* H V\right] Y.$$

Example

Dirac system with real potential f: $\quad H^{CS} = \begin{pmatrix} e^{-2\int_0^t f} & 0 \\ 0 & e^{-2\int_0^t f} \end{pmatrix}.$

de Branges' spaces of entire functions

Hardy space in $\mathbb{C}_+$:

$$H^2 = \{f \in Hol(\mathbb{C}_+) \mid ||f||^2_{H^2} = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx < \infty\}.$$

Notation: if $E(z)$ is entire we denote $E^{\#}(z) = \bar{E}(\bar{z})$.

Hermite-Biehler entire functions

An entire $E(z)$ is a Hermit-Biehler function ($E \in HB$) if

$$|E^{\#}(z)| < |E(z)|, \ z \in \mathbb{C}_+.$$

de Branges' space $B(E)$

If $E \in HB$ then $B(E)$ is defined as the space of entire functions F such that $F/E, \ F^{\#}/E \in H^2$. Hilbert structure: if $F, G \in B(E)$ then

$$< F, G >_{B(E)} = < F/E, G/E >_{H^2} = \int_{\mathbb{R}} F(x)\bar{G}(x)\frac{dx}{|E|^2}.$$

de Branges' spaces of entire functions: axiomatic definition

Theorem (de Branges)

Suppose that H is a Hilbert space of entire functions that satisfies

(A1) $F \in H, F(\lambda) = 0 \ \Rightarrow \ F(z)(z - \bar{\lambda})/(z - \lambda) \in H$ *with the same norm*

(A2) $\forall \lambda \notin \mathbb{R}$, *the point evaluation is bounded*

(A3) $F \to F^{\#}$ *is an isometry*

Then $H = B(E)$ *for some* $E \in HB$.

Examples of dB spaces

Example

E is a polynomial. $E \in HB \;\Leftrightarrow$ all zeros are in $\bar{\mathbb{C}}_-$. $B(E)$ consists of all poynomials of lesser degree.

Example

$E = e^{-iaz}$, $B(E) = PW_a$ (Payley-Wiener space).

Example

Let $\mu > 0$ be a finite measure on $\mathbb{R}$ such that polynomials are incomplete in $L^2(\mu)$. Then the closure of polynomials is a de Branges space.

Example

The same example with $\mathcal{E}_a = \{e^{ict},\ 0 \le c \le a\}$ in place of polynomials.

(What is E in the last two examples ???!!!)

Krein Systems meet de Branges' spaces

Let E be an Hermite-Biehler function. Put

$$A = (E + E^{\#})/2, \quad B = (E - E^{\#})/2i.$$

Reproducing kernels for $B(E)$: for any $\lambda \in \mathbb{C}$, $F \in B(E)$, $F(\lambda) =< F, K_\lambda >$ where

$$K_\lambda(z) = \frac{1}{\pi} \frac{B(z)\bar{A}(\lambda) - A(z)\bar{B}(\lambda)}{z - \bar{\lambda}}.$$

We will consider canonical systems

$$\Omega \dot{X}(t) = zH(t)X(t)$$

without "jump intervals", i.e. intervals where H is a constant matrix of rank 1.

Krein Systems meet de Branges' spaces

Solve a canonical system with any real initial condition at t_-. Denote the solution by $X_z(t) = (A_t(z), B_t(z))$.

Theorem

For any fixed t, $E_t(z) = A_t(z) - iB_t(z)$ *is a Hermit-Biehler entire function. The map* W *defined as* $WX_z = K^t_{\bar z}$ *extends unitarily to*

$$W : L^2(H, (t_-, t)) \to B(E_t)$$

(Weyl transform).

The formula for W*:*

$$Wf(z) = < Hf, X_{\bar z} >_{L^2(H,(t_-,t))} = \int_{t_-}^{t} < H(t)f(t), X_{\bar z}(t) > dt.$$

Examples of Weyl transforms

Krein- de Branges' theory:

$$\text{Canonical System on } (t_-, t_+) \overset{W}{\longleftrightarrow} B(E_t),\ t \in [t_-, t_+)$$

Example

Orthogonal polynomials satisfy difference equations corresponding to Krein systems with jump intervals. $B(E_t) = B_n$ is the same on each jump interval, $B_n = P_n$.

Example

Free Dirac ($Q = 0$): $E_t = e^{-2\pi izt}$, $B(E_t) = PW_t$ as sets.

Theorem

Let $B(E_t)$ be the chain of de Branges' spaces corresponding to a Dirac system with an L^1_{loc}-potential. Then $B(E_t) = PW_t$ as sets.

Gelfand-Levitan theory: a study of systems with $B(E_t) = PW_t$ as sets.

Let $B(E_t)$ be a chain of de Branges' spaces, $t \in [t_-, t_+)$ (the final space $B(E_{t_+})$ may or may not exist). There exists a locally finite positive measure μ on $\mathbb{R}$ such that

$$||f||_{B(E_t)} = ||f||_{L^2(\mu)} \text{ for all } f \in B(E_t) \text{ and all } t.$$

μ is the spectral measure for the corresponding Krein's system. Relation with de Branges' functions:

$$\frac{1}{|E_t|^2} \to \mu \text{ as } t \to t_+.$$

(In the limit circle case the limit will produce one of spectral measures.)

Consider the Dirac system

$$\Omega \dot{X} = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X - \begin{pmatrix} -q_2 & q_1 \\ -q_1 & q_2 \end{pmatrix} X$$

with potential $q = q_1 + iq_2$. Let $B(E_t) = PW_t$ (as sets) be the corresponding chain of de Branges' spaces. If K_0^t is the reproducing kernel for $B_t = B(E_t)$ then via the formula for the Weyl transform we get

$$\frac{d}{dt} K_0^t(0) = \frac{d}{dt} ||K_0^t||^2_{B_t} = E_t^2(0).$$

Recalling that $E_t = A_t - iB_t$, where $X_z(t) = (A_t(z), B_t(z))^T$ is a solution to the initial system, we obtain

$$\frac{d}{dt} E_t(0) = -qE_t(0)$$

or

$$q = -\frac{1}{2} \frac{d}{dt} \log E_t(0)^2 = -\frac{1}{2} \frac{d}{dt} \log \frac{d}{dt} ||K_0^t||^2$$

(Gelfand-Levitan formula).

We denote by $\hat{\mu}$ the Fourier transform of μ:

$$\hat{\mu}(z) = \int e^{-2\pi izt} d\mu(t).$$

Theorem (Krein)

Let q and μ be the potential and spectral measure of a Dirac system on $\mathbb{R}_+$. Then $q \in C(\mathbb{R}_+)$ iff $\hat{\mu} = \delta_0 + \phi$, where $\phi \in C(\mathbb{R})$.

Proof of the 'if' part:

For any $f \in B_t (\in PW_t)$

$$\int_{-t}^{t} \hat{f} = f(0) = < f, K_0^t >_{B_t} = \int f \bar{K}_0^t d\mu$$

if we put $\hat{K}_0^t = \psi^t$ then the last equation implies

$$1 = \psi^t * \hat{\mu} = \psi^t + \psi^t * \phi \quad \text{on} \quad [-t, t]$$

We obtained that the Fourier transform $\psi^t = \psi$ of K_0^t satisfies the Volterra equation

$$(I + \mathcal{K}_t)\psi = 1,$$

Where $\mathcal{K}_t$ is an operator on $L^2[-t,t]$, $\mathcal{K}_t f = f * \phi$. The operator $\mathcal{K}_t$ is an integral operator with a continuous kernel. Hence $\mathcal{K}_t$ is compact (approximate the kernel with polynomials). Hence $I + \mathcal{K}_t$ is Fredholm. Since

$$< (I + \mathcal{K}_t)f, g >_{L^2[-t,t]} = < f, g >_{L^2(\mu)} = < f, g >_{B_t},$$

$I + \mathcal{K}_t$ has a trivial kernel. Therefore, $I + \mathcal{K}_t$ is invertible and

$$\psi^t = (I + \mathcal{K}_t)^{-1} 1.$$

By the Fredholm-Hilbert Lemma on solutions of integral equations, $\psi^t(x)$ is differentiable with respect to t for each fixed $x \in [-t,t]$ and the derivative $\frac{d}{dt}\psi^t(x)$ is a continuous function of x.

Return to the de Branges' chain $B_t = B(E_t),\ E_t = A_t - iB_t$. Denote

$$\varepsilon^t = \hat{E}_t, \alpha^t = \hat{A}_t, \beta^t = \hat{B}_t.$$

WLOG we can assume that $E_t(0) > 0$ for all t. Then

$$B_t(z) = \frac{iz}{E_t(0)} K_0^t(z).$$

It follows that $\beta^t(x)$ is the x-derivative, in the sense of distributions, of a function $h_t(x) \in C[-t, t]$ that is continuous in x and continuously differentiable in t. Similar statements can be proved for α^t, ε^t.

We obtain that

$$E_t(0) = A_t(0) = \int_{-t}^{t} \alpha_t = f_t(t) - f_t(-t)$$

where $f_s(x)$ is a continuous function of x, continuously differentiable with respect to s. Notice, that since A is real, $\alpha_t(x) = \bar{\alpha}_t(-x)$ and $f_t(x) = \bar{f}_t(-x)$. Hence $f_t'(x) + \bar{f}_t'(-t)$, understood in the sense of distributions, is purely imaginary. Therefore

$$-q(t) = \frac{d}{dt} \log E_t(0) = \Re \frac{(\dot{f}_t(t) - \dot{f}_t(-t)) + (f_t'(t) + f_t'(-t))}{E_t(0)} =$$

$$\Re \frac{(\dot{f}_t(t) - \dot{f}_t(-t))}{E_t(0)}.$$

Since f_t is continuously differentiable in t, q is continuous. $\square$

Riemann zeta function

The Riemann ζ-function

$$\zeta(z) = \sum_{n=0}^{\infty} \frac{1}{n^z}$$

The Riemann ξ-function

$$\xi(z) = \frac{1}{2}\pi^{-z/2} z(z-1)\Gamma\left(\frac{z}{2}\right)\zeta(z).$$

ξ is entire satisfying

$$\xi(z) = \xi(1-z).$$

The zeros of the ξ-function are the non-trivial zeros of the ζ-function.

Put $A = \xi\left(\frac{1}{2} - iz\right),\ B = i\xi'\left(\frac{1}{2} - iz\right).$

Theorem (J. Lagarias, 2006)

The Riemann Hypothesis holds iff $E = A - iB$ is an Hermite-Biehler function.

Recall: $E \in HB \Leftrightarrow$ there exists a Krein Canonical System

$$\Omega\dot{X} = zH(t)X$$

generating E. Then the zeros of A, that are the zeros of the ζ function after $z \mapsto \frac{1}{2} - iz$, are the spectrum of the Krein Canonical System.

Two approaches to RH

Approach I

Construct a Hilbert space of entire functions, verify the axioms to prove that it is a de Branges' space, prove that the generating function is the desired $E(z)$.

Approach II

Construct a Hamiltonian $H(t)$ such that the corresponding Krein Canonical System generates $E(z)$ (Hilbert-Pólya operator).

Approach I: Mellin Transform, Sonine spaces

L. de Branges, J.-F. Burnol.

Consider two integral transforms on $L^2(\mathbb{R}_+)$:

The cosine (Fourier) transform

$$\mathcal{F}g(z) = 2\int_0^\infty \cos(2\pi tz)g(t)dt$$

The completed (right) Mellin transform

$$\mathcal{M}g(z) = \pi^{z/2}\Gamma\left(\frac{z}{2}\right)\int_0^\infty g(t)t^{-z}dx.$$

Consider a chain of subspaces $S_a \subset L^2(\mathbb{R}_+)$, $a > 0$ consisting of $f \in L^2(\mathbb{R}_+)$ such that $f = \mathcal{F}f = 0$ on $(0, a)$ (Sonine Spaces).

Approach I: Mellin Transform, Sonine spaces

Define the spaces

$$B_a = \mathcal{M}(S_a).$$

Then B_a form a de Branges chain of Hilbert spaces of entire functions [de Branges]. These spaces display "Riemann-type" behavior (order of growth, distribution of zeros [de Branges, Burnol]). For instance, reproducing kernels of B_a corresponding to the Riemann zeros form a complete system for all $a > 1$ and minimal system for all $a < 1$ [Burnol, 2004].

Approach II: Morse Potentials

J. Lagarias.

Consider the Schrödinger operator with the Morse potential:

$$-\frac{d^2}{dt^2} + V_k(t) \text{ on } [t_-, \infty), \quad V_k(t) = \frac{1}{4}e^{2t} + ke^t.$$

with a fixed boundary condition at t_-. Morse potentials arise in quantum physics (potentials for di-atomic molecules, magnetic fields on hyperbolic plane, Selberg trace formula) but are usually studied on the left half-axis or on the whole line.

On the right half-line, the spectrum is discrete, simple and bounded from below. The eigenvector corresponding to the spectral parameter λ is the Whittaker function $W_{k,\lambda}(t)$.

The Weyl asymptotics of the spectrum [Lagarias]:

$$\#\{\lambda_n < T\} = c_1\sqrt{T}\log T + c_2\sqrt{T} + O(1) \quad \text{as } T \to \infty.$$

Approach II: Morse Potentials

The entire function

$$F(z) = W_{k,z-\frac{1}{2}}(t)$$

displays Riemann-ξ behavior:

Theorem (Lagarias, 2009)

1) $F(z)$ is an entire function of order one and maximal type, real on $\mathbb{R}$ and on $\Re z = \frac{1}{2}$
2) $F(z) = F(1-z)$
3) (# of zeros in $[-T, T]$) $= \frac{2}{\pi} T \log T + \frac{2}{\pi}(2 \log 2 - 1 - \log t_{-}) T + O(1)$
4) All but finitely many zeros of F are on $\Re z = \frac{1}{2}$. All other zeros are on the real line. All zeros are simple, except possibly at $z = \frac{1}{2}$.