Embedding Theorems for weighted Bergman spaces and applications to Control Theory

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Outline

1. Linear Systems and infinite-time Admissibility
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2. Embedding Theorem for Bergman spaces
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2. Embedding Theorem for Bergman spaces
3. Applications
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2. Embedding Theorem for Bergman spaces
3. Applications
Linear Systems

System $S$

$A$
Linear Systems and infinite-time Admissibility

System $S$

input $u$  $\rightarrow$  $\rightarrow$  $\rightarrow$

$\leftarrow \downarrow A \uparrow$

output $y$  $\rightarrow$
Linear Systems und infinite-time Admissibility

System $S$

output $y$

$A$
Linear System $S$

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
x(0) &= x_0 \\
y(t) &= 
\end{aligned}
\]
### Linear System $\mathcal{S}$

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\text{exp stab. } C_0 \text{ semigroup } (T(t))_{t \geq 0} \\
x(0) &= x_0 \\
y(t) &= 
\end{align*}$$
Linear System $\mathcal{S}$

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\text{exp stab. } C_0 \text{ semigroup } (T(t))_{t \geq 0} \\
\in \mathcal{U} \text{ input space} \\
x(0) &= x_0 \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]
Linear System $\mathcal{S}$

\[
\begin{align*}
\dot{x}(t) &= A x(t) \\
&\quad \text{exp stab. } C_0 \text{ semigroup } \left( T(t) \right)_{t \geq 0} \\
&\quad \in \mathcal{U} \text{ input space}
\end{align*}
\]

\[
\begin{align*}
x(0) &= x_0 \\
y(t) &=
\end{align*}
\]
Linear System $S$

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
exp stab. $C_0$ semigroup $(T(t))_{t \geq 0} \in U$ input space

\[ x(0) = x_0 \in \mathcal{H} \text{ state space} \]

\[ y(t) = \]
Linear Systems and infinite-time Admissibility

Linear System $S$

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\text{exp stab. } C_0 \text{ semigroup } (T(t))_{t \geq 0} \\
U &\in \text{input space}
\end{align*}
\]

\[
\begin{align*}
x(0) &= x_0 \\
H &\in \text{state space}
\end{align*}
\]

\[
\begin{align*}
y(t) &= Cx(t) + Du(t) \\
\end{align*}
\]
Linear System $S$

$$\begin{align*}
\dot{x}(t) &= Ax(t) \\
&\text{exp stab. } C_0 \text{ semigroup } (T(t))_{t \geq 0} \\
\end{align*}$$

\begin{align*}
\begin{align*}
\dot{x}(0) &= x_0 \\
&\in \mathcal{H} \text{ state space} \\
y(t) &= Cx(t)
\end{align*}
\end{align*}$$
**Linear System**

### Linear System $\mathcal{S}$

- **States Equation:**
  \[
  \dot{x}(t) = Ax(t) \\
  \text{exp stab. } C_0 \text{ semigroup } (T(t))_{t \geq 0}
  \]

- **Initial Condition:**
  \[
  x(0) = x_0 \\
  \in \mathcal{H} \text{ state space}
  \]

- **Output Equation:**
  \[
  y(t) = Cx(t)
  \]

**Assumption:**
- $(T(t))$ is a normal semigroup (or eigenvectors $(\phi_n)$ of $A$ form ONB of $\mathcal{H}$)
Definition

\((A, C)\) is (infinite-time) admissible, if

\[
\int_0^\infty |Cx(t)|^2 dt \lesssim \|x_0\|^2
\]

for all initial values \(x_0 \in \mathcal{H}\)
(A, C) is (infinite-time) admissible, if

\[ \int_0^\infty |Cx(t)|^2 dt \lesssim \|x_0\|^2 \]

for all initial values \(x_0 \in \mathcal{H}\)

(can be thought of as a well-posedness property)
Weiss’ Theorem

Theorem (Weiss ’92)

Let $A$ be the generator of a normal exponentially stable $C_0$ semigroup and $C$ an observation operator. Then TFAE:

1. $(A, C)$ is admissible

2. 
   
   $$\|C(z - A)^{-1}\| \lesssim \frac{1}{\Re z} \quad (z \in \mathbb{C}_+).$$


Let $x_0 = \sum_{n=0}^{\infty} \alpha_n \phi_n$. Then $x(t) = \sum_{n=0}^{\infty} \alpha_n e^{-\lambda_n t} \phi_n$, and

$$\int_0^{\infty} Cx(t)\overline{f(t)} dt = \sum_{n=0}^{\infty} C\phi_n \alpha_n \langle e^{-\lambda_n t}, f(t) \rangle = \sum_{n=0}^{\infty} C\phi_n \alpha_n \overline{Lf(\lambda_n)}.$$
Proof I (for diagonal $A$)

Let $x_0 = \sum_{n=0}^{\infty} \alpha_n \phi_n$. Then $x(t) = \sum_{n=0}^{\infty} \alpha_n e^{-\lambda_n t} \phi_n$, and

$$\int_0^{\infty} Cx(t) \overline{f(t)} dt = \sum_{n=0}^{\infty} C\phi_n \alpha_n \langle e^{-\lambda_n t}, f(t) \rangle = \sum_{n=0}^{\infty} C\phi_n \alpha_n \mathcal{L}f(\lambda_n).$$

Thus $(A, C)$ is admissible, if and only if

$$\sum_{n=0}^{\infty} |C\phi_n|^2 |\mathcal{L}f(\lambda_n)|^2 \lesssim \|f\|_2^2$$

$$\Leftrightarrow \mu = \sum_{n=0}^{\infty} |C\phi_n|^2 \delta_{\lambda_n} \text{ is a Carleson measure}.$$
Let \( x_0 = \sum_{n=0}^{\infty} \alpha_n \phi_n \). Then \( x(t) = \sum_{n=0}^{\infty} \alpha_n e^{-\lambda_n t} \phi_n \), and

\[
C(z - A)^{-1} x_0 = \sum_{n=0}^{\infty} C\phi_n \alpha_n \frac{1}{z + \lambda_n},
\]
Proof II

Let \( x_0 = \sum_{n=0}^{\infty} \alpha_n \phi_n \). Then \( x(t) = \sum_{n=0}^{\infty} \alpha_n e^{-\lambda_n t} \phi_n \), and

\[
C(z - A)^{-1} x_0 = \sum_{n=0}^{\infty} C \phi_n \alpha_n \frac{1}{z + \lambda_n},
\]

Thus the resolvent condition (2) holds, if and only if

\[
\sum_{n=0}^{\infty} |C \phi_n|^2 \left| \frac{1}{z + \lambda_n} \right|^2 \lesssim \frac{1}{\Re z}
\]

\[\iff \sum_{n=0}^{\infty} |C \phi_n|^2 \delta \lambda_n \text{ is a Carleson measure}.\]

\[\]
\(\alpha\)-Admissibility

**Definition**

Let \( \alpha > -1 \). \((A, C)\) is \(\alpha\)-admissible, if

\[
\int_0^\infty t^\alpha |Cx(t)|^2 dt \lesssim \|x_0\|^2
\]

for all initial values \(x_0 \in \mathcal{H}\).

**Theorem (Haak, Le Merdy 2005)**

Let \( \alpha > 0 \) and let \( A \) be the generator of a diagonal exponentially stable \(C_0\) semigroup and \( C \) an observation operator. Then TFAE:

1. \((A, C)\) is \(\alpha\)-admissible
2. 

\[
\|C(z - A)^{-1-\alpha}\| \lesssim \frac{1}{(\Re z)^{1+\alpha}} \quad (z \in \mathbb{C}_+).
\]
Further Generalisations

Results for $\alpha < 0$ are due to Wynn, results for Lebesgue measure on $\mathbb{R}^+$ positive measures of finite support are due to Harper.
Further Generalisations

Results for $\alpha < 0$ are due to Wynn, results for Lebesgue measure on $\mathbb{R}^+$ and positive measures of finite support are due to Harper.

Question

What is the most general class of weights $w(t)$ on $\mathbb{R}^+$, such that $w$-admissibility can be characterised by a resolvent-type condition?
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For a positive Borel measure \( \nu \) on \( \overline{\mathbb{C}_+} \), let

\[
A^2_\nu = \{ f : \mathbb{C}_+ \to \mathbb{C} \text{ hol.} : \sup_{\varepsilon > 0} \| f(\cdot + \varepsilon) \|_{L^2_\nu} < \infty \}.
\]

**Question**

What is the most general class of “radial” positive Borel measures \( \nu = \tilde{\nu} \otimes d\lambda \) on \( \overline{\mathbb{C}_+} \), such that embeddings

\[
A^2_\nu(\overline{\mathbb{C}_+}) \hookrightarrow L_\mu(\mathbb{C}_+)
\]

can be tested on (derivatives of) reproducing kernels?

(Results by Oleinek, Pau, Pelaez, Rättyä...)
Definition

Let $\nu = \tilde{\nu} \otimes d\lambda$ be a positive Borel measure on $\overline{C_+}$. We say that $A^2_\nu(C_+)$ is a Zen space, if the following $(\Delta_2)$-condition holds:

$$\sup_{r>0} \frac{\tilde{\nu}([0, 2r))}{\tilde{\nu}([0, r))} < \infty.$$
The embedding theorem

Theorem (Jacob, Partington, P.)

Let $1 \leq p < \infty$, $A^p_\nu(C_+) \, \text{be a Zen space and let } \mu \, \text{be a positive Borel measure on } C_+$. Then TFAE:

1. The embedding

$$A^p_\nu(C_+) \hookrightarrow L^p_\mu(C_+)$$

is bounded.

2. $\mu(Q_l) \lesssim \nu(Q_l)$ for all intervals $l \subset i\mathbb{R}$

3. For a suitably large $N \in \mathbb{N}$,

$$\left\| \frac{1}{(z + \bar{w})^N} \right\|_{L^p_\mu} \lesssim \left\| \frac{1}{(z + \bar{w})^N} \right\|_{A^p_\nu} \quad \text{for all } w \in C_+$$
Embedding Theorem for Bergman spaces

Proof - for the standard Bergman space $A^2$

For $n \in \mathbb{Z}$, let $\mu_n$ be the restriction of the measure $\mu$ to the vertical strip

$$E_n = \{ z \in \mathbb{C}_+ : 2^{n+1} \leq \Re \xi < 2^{n+1} \}.$$

Clearly $\mu = \sum_{n=-\infty}^{\infty} \mu_n$.

Lemma

Let $\mu$ be a regular Borel measure on $\mathbb{C}_+$ with $M = \sup_{l \text{ interval}} \frac{\mu(Q_l)}{|l|^2} < \infty$. For each $n \in \mathbb{Z}$, let $\mu_n$ be the restriction of the measure $\mu$ to the vertical strip $E_n$.

Then for each $n$, $\mu_n$ is a Carleson measure for the shifted half plane $\mathbb{C}_{2^n}$, with Carleson constant

$$C(2^n, \mu_n) \leq 4^2 2^n M.$$
Lemma

Let \( f \in A^2 \). Then

\[
\| f \|_{A^2}^2 \approx \sum_{n=-\infty}^{\infty} 2^n \| f \|_{H^2(\mathbb{C}^n)}^2
\]

Hence

\[
\int_{\mathbb{C}^+} |f(z)|^2 d\mu(z) = \sum_{n=-\infty}^{\infty} \int_{\mathbb{C}^{2n}} |f(z)|^2 d\mu_n(z)
\]

\[
\leq M \sum_{n=-\infty}^{\infty} 4^2 2^n \| f \|_{H^2_n}^2 \lesssim \| f \|_{A^2}^2.
\]
It is not difficult to see that we can reduce general Zen spaces to the case

\[ \tilde{\nu} = \sum_{n=-\infty}^{\infty} \beta_n \delta_{a_n}. \]

The key part is now to slice up the measure \( \mu \) in an appropriate way:
Let $N \in \mathbb{Z}$ and suppose that $\mu$ is supported on the closed half-plane $\overline{\mathbb{C}_{a_N}}$. Then there exist positive regular Borel measures $\mu_n$, $n \geq N$, such that

1. $\mu = \sum_{n=N}^{\infty} \mu_n$;
2. $\mu_n$ is supported on the closed half-plane $\overline{\mathbb{C}_{a_n}}$;
3. There exists a constant $C'_\beta > 0$ such that for all intervals $I \subset \mathbb{R}$, $\mu_n(Q_I) \leq C'_\beta \nu_n(Q_I)$ ($n > N$), $\mu_N(Q_I) \leq C'_\beta \sum_{k=-\infty}^{N-1} \beta_k$.

Moreover, $\mu_n$ is a Carleson measure for the shifted half plane $\mathbb{C}_{a_n-1}$, with Carleson constant $C_{a_n-1}(\mu_n) \leq C'_c \beta_n$ ($n > N$), $C_{a_N-1}(\mu_N) \leq C'_c \sum_{k=-\infty}^{N-1} \beta_k$, where $c$ is implicit in the definition of the sequence $(a_n)$. 

Key Lemma

Let \( N \in \mathbb{Z} \) and suppose that \( \mu \) is supported on the closed half-plane \( \overline{C_{a_N}} \). Then there exist positive regular Borel measures \( \mu_n, n \geq N \), such that

1. \( \mu = \sum_{n=N}^{\infty} \mu_n \);
Key Lemma

Let $N \in \mathbb{Z}$ and suppose that $\mu$ is supported on the closed half-plane $\overline{C_{a_N}}$. Then there exist positive regular Borel measures $\mu_n, n \geq N$, such that

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2. $\mu_n$ is supported on the closed half-plane $\overline{C_{a_n}}$;
Key Lemma

Let $N \in \mathbb{Z}$ and suppose that $\mu$ is supported on the closed half-plane $\overline{C_{a_N}}$. Then there exist positive regular Borel measures $\mu_n$, $n \geq N$, such that

1. $\mu = \sum_{n=N}^{\infty} \mu_n$;
2. $\mu_n$ is supported on the closed half-plane $\overline{C_{a_n}}$;
3. There exists a constant $C' > 0$ such that for all intervals $I \subset i\mathbb{R}$, $\mu_n(Q_I) \leq C' \nu_n(Q_I)$ ($n > N$), $\mu_N(Q_I) \leq C' \sum_{k=-\infty}^{N} \nu_k(Q_I)$.
Key Lemma

Let $N \in \mathbb{Z}$ and suppose that $\mu$ is supported on the closed half-plane $\overline{C_{a_N}}$. Then there exist positive regular Borel measures $\mu_n$, $n \geq N$, such that

1. $\mu = \sum_{n=N}^{\infty} \mu_n$;
2. $\mu_n$ is supported on the closed half-plane $\overline{C_{a_n}}$;
3. There exists a constant $C' > 0$ such that for all intervals $I \subset i\mathbb{R}$,
   $$\mu_n(Q_I) \leq C' \nu_n(Q_I) \quad (n > N), \quad \mu_N(Q_I) \leq C' \sum_{k=-\infty}^{N} \nu_k(Q_I).$$

Moreover, $\mu_n$ is a Carleson measure for the shifted half plane $\overline{C_{a_{n-1}}}$, with Carleson constant

$$C_{a_{n-1}}(\mu_n) \leq \frac{C'}{c} \beta_n \quad (n > N), \quad C_{a_{N-1}}(\mu_N) \leq \frac{C'}{c} \sum_{k=-\infty}^{N} \beta_k,$$

where $c$ is implicit in the definition of the sequence $(a_n)$. 

Sandra Pott (Lund)
A “slicewise” application of the Carleson Embedding Theorem now finishes the proof.
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Regular measures $\nu$

**Definition**

Let $\nu = \tilde{\nu} \otimes d\lambda$ be a positive Borel measure on $\overline{\mathbb{C}}_+$. We say that $\nu$ is regular, if

\[
\sup_{r > 0} \frac{\tilde{\nu}([0, 2r))}{\tilde{\nu}([0, r))} < \infty,
\]

\[
\sup_{M > 0} \inf_{r > 0} \frac{\tilde{\nu}([0, Mr))}{\tilde{\nu}([0, r))} > 1.
\]
Let $A^2_\nu$ be a Zen space, $d_\nu = d\tilde{\nu} \otimes d\lambda$, with $\tilde{\nu}$ regular. Let $b : \mathbb{C}_+ \to \mathbb{C}$ be in the Bloch space $\mathcal{B}(\mathbb{C}_+)$. Then the little Hankel operator

$$A^2_\nu \rightarrow \overline{A^2_\nu}, \quad f \mapsto Q_\nu \overline{bf}$$

is bounded.

Here, $Q_\nu$ denotes the orthogonal projection $Q_\nu : L^2(\mathbb{C}_+, d_\nu) \to \overline{A^2_\nu}$. 
Integral Operators

For $g : \mathbb{C}_+ \rightarrow \mathbb{C}$ analytic, consider the integral operator

$$f \mapsto T_g f(z) = \int_1^z g'(s)f(s)ds.$$ 

Theorem

Let $A^2_\nu$ be a Zen space with $\tilde{\nu}$ regular. Let $g : \mathbb{C}_+ \rightarrow \mathbb{C}$ be analytic. Then the integral operator

$$A^2_\nu \rightarrow A^2_\nu, \quad f \mapsto T_g f$$

is bounded, if and only if $g \in \mathcal{B}(\mathbb{C}_+)$. 