Embedding Theorems for weighted Bergman spaces and applications to Control Theory

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Outline

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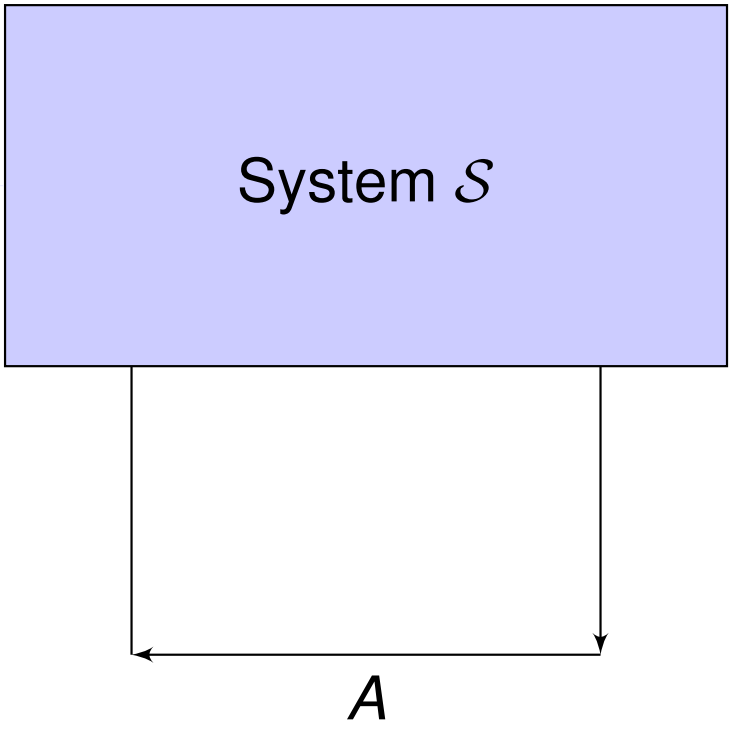
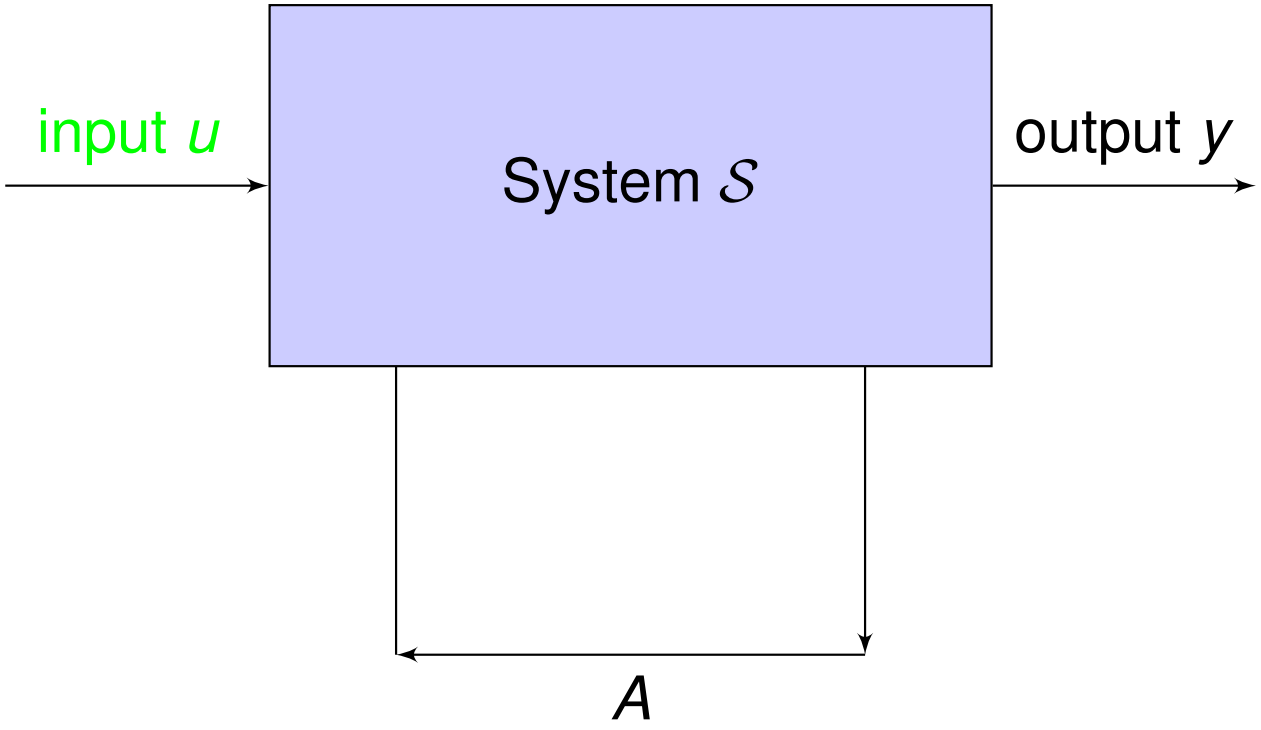
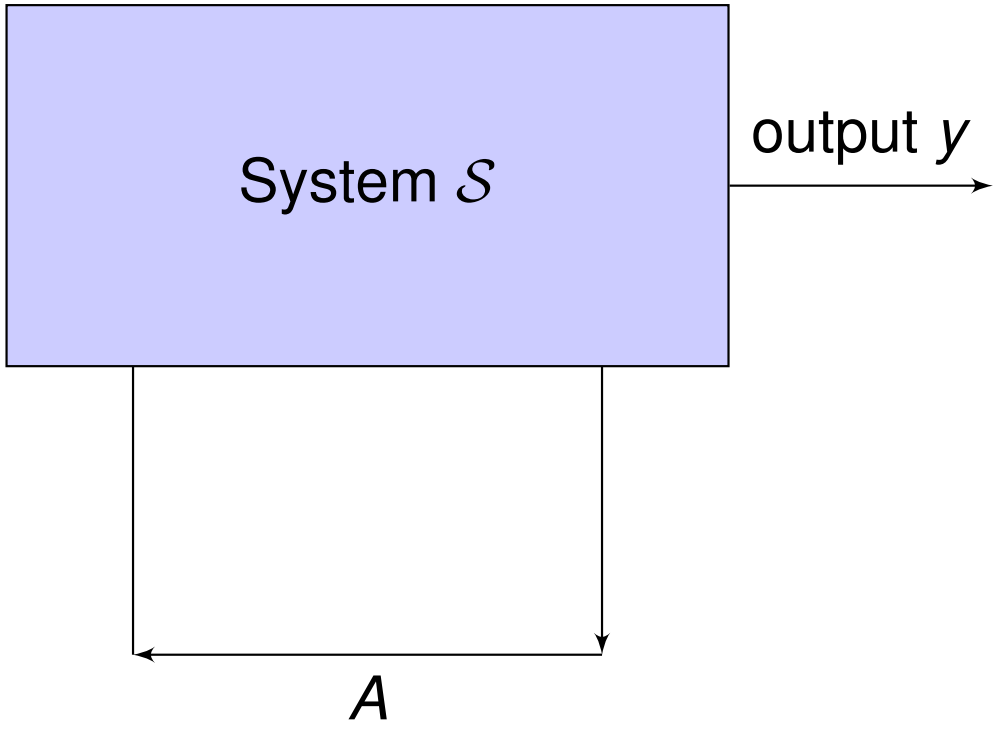
Linear Systems



Linear Systems and infinite-time Admissibility



Linear Systems und infinite-time Admissibility



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$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \\ y(t) &= \end{aligned}$$

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Assumption:

- $(T(t))$ is a normal semigroup (or eigenvectors (ϕ_n) of A form ONB of $\mathcal{H}$)

Infinite-time Admissibility

Definition

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(can be thought of as a well-posedness property)

Weiss' Theorem

Theorem (Weiss '92)

Let A be the generator of a normal exponentially stable C_0 semigroup and C an observation operator. Then TFAE:

1. (A, C) *is admissible*
2. $$\|C(z-A)^{-1}\| \lesssim \frac{1}{\Re z} \quad (z \in \mathbb{C}_+).$$

Proof I (for diagonal A)

Let $x_0 = \sum_{n=0}^{\infty} \alpha_n \phi_n$. Then $x(t) = \sum_{n=0}^{\infty} \alpha_n e^{-\lambda_n t} \phi_n$, and

$$\int_0^{\infty} Cx(t)\bar{f}(t)dt = \sum_{n=0}^{\infty} C\phi_n \alpha_n \langle e^{-\lambda_n t}, f(t)\rangle = \sum_{n=0}^{\infty} C\phi_n \alpha_n \overline{\mathcal{L}f(\lambda_n)}.$$

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Thus (A, C) is admissible, if and only if

$$\sum_{n=0}^{\infty} |C\phi_n|^2 |\mathcal{L}f(\lambda_n)|^2 \lesssim \|f\|_2^2$$

$$\Leftrightarrow \quad \mu = \sum_{n=0}^{\infty} |C\phi_n|^2 \delta_{\lambda_n} \text{ is a Carleson measure .}$$

Proof II

Let $x_0 = \sum_{n=0}^{\infty} \alpha_n \phi_n$. Then $x(t) = \sum_{n=0}^{\infty} \alpha_n e^{-\lambda_n t} \phi_n$, and

$$C(z-A)^{-1}x_0 = \sum_{n=0}^{\infty} C\phi_n \alpha_n \frac{1}{z+\lambda_n},$$

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Thus the resolvent condition (2) holds, if and only if

$$\sum_{n=0}^{\infty} |C\phi_n|^2 \left| \frac{1}{z+\lambda_n} \right|^2 \lesssim \frac{1}{\Re z}$$

$$\Leftrightarrow \quad \sum_{n=0}^{\infty} |C\phi_n|^2 \delta_{\lambda_n} \text{ is a Carleson measure .}$$

α-Admissibility

Definition

Let $\alpha > -1$. (A, C) is α-admissible, if

$$\int_0^\infty t^\alpha |Cx(t)|^2 dt \lesssim \|x_0\|^2$$

for all initial values $x_0 \in \mathcal{H}$.

Theorem (Haak, Le Merdy 2005)

Let $\alpha > 0$ and let A be the generator of a diagonal exponentially stable C_0 semigroup and C an observation operator. Then TFAE:

1. *(A, C) is α-admissible*
2. $$\|C(z-A)^{-1-\alpha}\| \lesssim \frac{1}{(\Re z)^{1+\alpha}} \quad (z \in \mathbb{C}_+).$$

Further Generalisations

Results for $\alpha < 0$ are due to Wynn, results for Lebesgue measure on $\mathbb{R}^+$ + positive measures of finite support are due to Harper.

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Question

What is the most general class of weights $w(t)$ on $\mathbb{R}_+$, such that w-admissibility can be characterised by a resolvent-type condition?

Outline

Translation to Bergman spaces

For a positive Borel measure ν on $\overline{\mathbb{C}_+}$, let

$$A^2_\nu = \{f : \mathbb{C}_+ \to \mathbb{C} \text{ hol.} : \sup_{\varepsilon>0} \|f(\cdot + \varepsilon)\|_{L^2_\nu} < \infty\}.$$

Question

What is the most general class of "radial" positive Borel measures $\nu = \tilde{\nu} \otimes d\lambda$ on $\overline{\mathbb{C}_+}$, such that embeddings

$$A^2_\nu(\overline{\mathbb{C}_+}) \hookrightarrow L_\mu(\mathbb{C}_+)$$

can be tested on (derivatives of) reproducing kernels ?

(Results by Oleinek, Pau, Pelaez, Rättyä...)

Zen spaces

Definition

Let $\nu = \tilde{\nu} \otimes d\lambda$ be a positive Borel measure on $\overline{\mathbb{C}_+}$. We say that $A^2_\nu(\mathbb{C}_+)$ is a Zen space, if the following (Δ_2)-condition holds:

$$\sup_{r>0} \frac{\tilde{\nu}([0,2r))}{\tilde{\nu}([0,r))} < \infty.$$

The embedding theorem

Theorem (Jacob, Partington, P.)

Let $1 \leq p < \infty$, $A^p_\nu(\mathbb{C}_+)$ *be a Zen space and let* μ *be a positive Borel measure on* $\mathbb{C}_+$. *Then TFAE:*

1. *The embedding*
$$A^p_\nu(\overline{\mathbb{C}_+}) \hookrightarrow L^p_\mu(\mathbb{C}_+)$$
is bounded.
2. $$\mu(Q_I) \lesssim \nu(Q_I) \quad \text{for all intervals } I \subset i\mathbb{R}$$
3. *For a suitably large* $N \in \mathbb{N}$,
$$\left\| \frac{1}{(z+\bar{w})^N} \right\|_{L^p_\mu} \lesssim \left\| \frac{1}{(z+\bar{w})^N} \right\|_{A^p_\nu} \quad \text{for all } w \in \mathbb{C}_+$$

Proof - for the standard Bergman space A^2

For $n \in \mathbb{Z}$, let μ_n be the restriction of the measure μ to the vertical strip

$$E_n = \{z \in \mathbb{C}_+ : 2^{n+1} \leq \Re\xi < 2^{n+1}\}.$$

Clearly $\mu = \sum_{n=-\infty}^{\infty} \mu_n$.

Lemma

Let μ be a regular Borel measure on $\mathbb{C}_+$ with $M = \sup_{I \text{ interval}} \frac{\mu(Q_I)}{|I|^2} < \infty$. For each $n \in \mathbb{Z}$, let μ_n be the restriction of the measure μ to the vertical strip E_n.
Then for each n, μ_n is a Carleson measure for the shifted half plane $\mathbb{C}_{2^n}$, with Carleson constant

$$\mathcal{C}(2^n, \mu_n) \leq 4^2 2^n M.$$

More on the standard Bergman proof

Lemma

Let $f \in A^2$. Then

$$\|f\|_{A^2}^2 \approx \sum_{n=-\infty}^{\infty} 2^n \|f\|_{H^2(\mathbb{C}_{2^n})}^2$$

Hence

$$\begin{aligned} \int_{\mathbb{C}_+} |f(z)|^2 d\mu(z) &= \sum_{n=-\infty}^{\infty} \int_{\mathbb{C}_{2^n}} |f(z)|^2 d\mu_n(z) \\ &\leq M \sum_{n=-\infty}^{\infty} 4^2 2^n \|f\|_{H^2_{2^n}} \lesssim \|f\|_{A^2}^2. \end{aligned}$$

Back to reality

It is not difficult to see that we can reduce general Zen spaces to the case

$$\tilde{\nu} = \sum_{n=-\infty}^{\infty} \beta_n \delta_{a_n}.$$

The key part is now to slice up the measure μ in an appropriate way:

Key Lemma

Let $N \in \mathbb{Z}$ and suppose that μ is supported on the closed half-plane $\overline{\mathbb{C}_{a_N}}$. Then there exist positive regular Borel measures μ_n, $n \geq N$, such that

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1. $\mu = \sum_{n=N}^{\infty} \mu_n$;
2. μ_n is supported on the closed half-plane $\overline{\mathbb{C}_{a_n}}$;
3. There exists a constant $C' > 0$ such that for all intervals $I \subset i\mathcal{R}$, $\mu_n(Q_I) \leq C'\nu_n(Q_I) \quad (n > N), \quad \mu_N(Q_I) \leq C' \sum_{k=-\infty}^{N} \nu_k(Q_I)$.

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Moreover, μ_n is a Carleson measure for the shifted half plane $\mathbb{C}_{a_{n-1}}$, with Carleson constant

$$\mathcal{C}_{a_{n-1}}(\mu_n) \leq \frac{C'}{c}\beta_n \quad (n > N), \quad \mathcal{C}_{a_{N-1}}(\mu_N) \leq \frac{C'}{c} \sum_{k=-\infty}^{N} \beta_k,$$

where c is implicit in the definition of the sequence (a_n).

Finishing off

A “slicewise” application of the Carleson Embedding Theorem now finishes the proof. □

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Regular measures ν

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Let $\nu = \tilde{\nu} \otimes d\lambda$ be a positive Borel measure on $\overline{\mathbb{C}_+}$. We say that ν is regular, if

$$\sup_{r>0} \frac{\tilde{\nu}([0,2r))}{\tilde{\nu}([0,r))} < \infty,$$

$$\sup_{M>0} \inf_{r>0} \frac{\tilde{\nu}([0,Mr))}{\tilde{\nu}([0,r))} > 1.$$

Little Hankel Operators

Theorem

Let A^2_ν be a Zen space, $d\nu = d\tilde{\nu} \otimes d\lambda$, with $\tilde{\nu}$ regular. Let $b : \mathbb{C}_+ \to \mathbb{C}$ be in the Bloch space $\mathcal{B}(\mathbb{C}_+)$. Then the little Hankel operator

$$A^2_\nu \to \overline{A^2_\nu}, \quad f \mapsto Q_\nu \bar{b} f$$

is bounded.
Here, Q_ν denotes the orthogonal projection $Q_\nu : L^2(\mathbb{C}_+, d\nu) \to \overline{A^2_\nu}$.

Integral Operators

For $g : \mathbb{C}_+ \to \mathbb{C}$ analytic, consider the integral operator

$$f \mapsto T_g f(z) = \int_1^z g'(s) f(s) ds.$$

Theorem

Let A^2_ν be a Zen space with $\tilde{\nu}$ regular. Let $g : \mathbb{C}_+ \to \mathbb{C}$ be analytic. Then the integral operator

$$A^2_\nu \to A^2_\nu, \quad f \mapsto T_g f$$

is bounded, if and only if $g \in \mathcal{B}(\mathbb{C}_+)$.