The Neumann-Poincaré operator on domains with corners

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Based on a recent joint work with

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and not so recent work with

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Layer potentials

Classically related to physical processes and used to produce harmonic functions with prescribed boundary data.

Not yet tired of new applications and surprising theoretical turns.

Sobolev space

$\Omega \subset \mathbb{R}^n$, $n \geq 2$, open Lipschitz domain with connected boundary.

$H^1(\Omega)$ consists of all $V \in L^2(U)$ such that

$$\|V\|^2_{H^1(U)} = \|V\|^2_{L^2(U)} + \|\nabla V\|^2_{L^2(U)} < \infty.$$

Boundary spaces

Similarly $H^1(\partial\Omega)$.

For $0 < s < 1$, we obtain $H^s(\partial U)$ on the real interpolation scale between $L^2(\partial\Omega)$ and $H^1(\partial\Omega)$.

Alternatively $H^s(\partial\Omega)$ is a Besov space,

$$\|v\|^2_{H^s(\partial\Omega)} \sim \|v\|^2_{L^2(\partial\Omega)} + \int_{\partial\Omega\times\partial\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{n-1+2s}}\, d\sigma(x)\, d\sigma(y),$$

where σ denotes $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$.

We define $H^{-s}(\partial\Omega)$, $0 \le s \le 1$, as the dual of $H^s(\partial\Omega)$ under the (sesquilinear) L^2-pairing

Neumann-Poincaré operator

Fundamental solution for Laplace operator

$$G(x,y) = \begin{cases} -\omega_n^{-1} \log|x-y|, & n = 2, \\ \omega_n^{-1}|x-y|^{2-n}, & n \geq 3, \end{cases}$$

where ω_n is the measure of the unit sphere in $\mathbb{R}^n$.

By the *Neumann-Poincaré operator*, or the *boundary double layer potential*, $K : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ we mean the operator

$$Kf(x) = -2 \int_{\partial\Omega} \partial_{n_y} G(x,y) f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

where n_y denotes the outward normal derivative at y.

Boundedness of K

$K : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$ is bounded [Coifman, McIntosh and Meyer 1982], [Verchota 1984].

$K : H^{1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ bounded by symmetry and interpolation arguments.

$K^* : H^{-1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ is our main character.

Spatial layer potentials

For $x \notin \partial\Omega$ we write

$$Df(x) = \int_{\partial\Omega} \partial_{n_y} G(x,y) f(y)\, d\sigma(y), \quad x \notin \partial\Omega,$$

and call D the *double layer potential*. For $g \in H^{-1/2}(\partial\Omega)$ the *single layer potential* S is defined by

$$Sg(x) = \int_{\partial\Omega} G(x,y) f(y)\, d\sigma(y), \quad x \in \mathbb{R}^n.$$

Energy norm

Denote the exterior of Ω by $\Omega_e = \overline{\Omega}^c$ and by $\mathfrak{H}$ the space of harmonic functions h on $\Omega \cup \Omega_e$ with $\lim_{x\to\infty} h(x) = 0$ and finite energy,

$$\|h\|_{\mathfrak{H}}^2 = \int_{\Omega\cup\Omega_e} |\nabla h|^2 \, dx < \infty.$$

To ensure that $\mathfrak{H}$ is a Hilbert space we also require that if $h \neq 0$ and $h_e = h|_{\Omega_e} = 0$, then $h_i = h|_{\Omega}$ is non-constant.

Traces

Each element $h \in \mathfrak{H}$ has an interior trace $\mathrm{Tr}_{\mathrm{int}}\, h = \mathrm{Tr}\, h_i \in H^{1/2}(\partial\Omega)$ and an exterior trace $\mathrm{Tr}_{\mathrm{ext}}\, h = \mathrm{Tr}\, h_e \in H^{1/2}(\partial\Omega)$.

By the classical Poincaré inequality for bounded Lipschitz domains U and the fact that the trace $\mathrm{Tr} : H^1(U) \to H^{1/2}(\partial U)$ is continuous, we see that the interior and the exterior traces are continuous as maps from $\mathfrak{H}$ to $H^{1/2}(\partial\Omega)$.

The interior and exterior trace normal derivatives $\partial_n^{\mathrm{int}} h, \partial_n^{\mathrm{ext}} h \in H^{-1/2}(\partial\Omega)$ are defined by duality and via Green's formula.

Orthogonal decompositions

One interprets an element $h \in \mathfrak{H}$ as a pair $(h_i, h_e) = (h|_\Omega, h|_{\Omega_e})$, with the corresponding orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_i \oplus \mathfrak{H}_e$. We denote by P_i and P_e the orthogonal projections onto $\mathfrak{H}_i$ and $\mathfrak{H}_e$ respectively, so that $P_i(h_i, h_e) = (h_i, 0)$ and $P_e(h_i, h_e) = (0, h_e)$.

Second orthogonal decomposition: let

$$\mathfrak{S} = \{h \in \mathfrak{H} \,:\, \mathrm{Tr}_{\mathrm{int}}\, h = \mathrm{Tr}_{\mathrm{ext}}\, h\} = S(H^{-1/2}(\partial\Omega))$$

denote the space of single layer potentials in $\mathfrak{H}$, and let

$$\mathfrak{D} = \{h \in \mathfrak{H} \,:\, \partial_n^{\mathrm{int}} h = \partial_n^{\mathrm{ext}} h\} = D(H_0^{1/2}(\partial\Omega))$$

denote the space of double layer potentials. Then $\mathfrak{H} = \mathfrak{S} \oplus \mathfrak{D}$ and we write P_s and P_d for the corresponding projections.

Jump formulae

For $f \in H^{-1/2}(\partial\Omega)$ ($f \perp 1$ if $n = 2$) and $g \in H_0^{1/2}(\partial\Omega)$, arguing with smooth functions and the continuity of operators involved, the well known jump formulae for S and K take on the form

$$
\begin{aligned}
\mathrm{Tr}_{\mathrm{int}}\, Sf &= \mathrm{Tr}_{\mathrm{ext}}\, Sf = Sf|_{\partial\Omega}, & \partial_n^{\mathrm{int}} Sf &= \frac{1}{2}(f - K^* f),\\
\partial_n^{\mathrm{ext}} Sf &= \frac{1}{2}(-f - K^* f), & \mathrm{Tr}_{\mathrm{int}}\, Dg &= \frac{1}{2}(-g - Kg),\\
\mathrm{Tr}_{\mathrm{ext}}\, Dg &= \frac{1}{2}(g - Kg), & \partial_n^{\mathrm{int}} Dg &= \partial_n^{\mathrm{ext}} Dg.
\end{aligned}
$$

Classical boundary problems as integral equations

Solve $\Delta u = 0$ in Ω with $\operatorname{Tr} u = \phi$ by searching

$$u = Dg, \quad g \in H_0^{1/2}(\partial\Omega).$$

And similarly $\Delta v = 0$ in Ω with $\partial_n v = \psi$

as

$$v = Sf, \quad f \in H^{-1/2}(\partial\Omega).$$

Hence the need to compute $\sigma(K|H_0 1/2)$ and study the convergence of the Neumann series for $(\lambda - K)^{-1}$.

Poincaré variational problem

Study the balance of energies

$$\frac{\|\nabla Sg\|_e^2 - \|\nabla Sg\|_i^2}{\|\nabla Sg\|_e^2 + \|\nabla Sg\|_i^2}$$

as a variational problem in $g \in H_0^{-1/2}(\partial\Omega)$.

Theorem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open, bounded Lipschitz domain with connected boundary. The operator $K^* : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is similar to the angle operator $P_s(P_e - P_i)P_s : \mathfrak{S} \longrightarrow \mathfrak{S}$.

Smooth boundary

In this case the successive singular values

$$\sigma_k = \max_{g \perp \{g_0, \ldots, g_{k-1}\}} \frac{\|\nabla Sg\|_e^2 - \|\nabla Sg\|_i^2}{\|\nabla Sg\|_e^2 + \|\nabla Sg\|_i^2}$$

are attained at $g = g_k$ and the sequence $g_0, g_1, g_{-1}, g_2, g_{-2}, \ldots$ is complete in $H_0^{-1/2}$.

Proof based on Plemelj intertwining formula

$$DS = SD^*$$

and the theory of symmetrizable operators (Carleman, Korn, M.G. Krein).

Angle operator

$K^* : H_0^{-1/2}(\partial\Omega) \to H_0^{-1/2}(\partial\Omega)$ can be regarded in two distinct ways as the angle operator between the orthogonal decompositions

$$\mathfrak{H} = \mathfrak{H}_i \oplus \mathfrak{H}_e = \mathfrak{S} \oplus \mathfrak{D}.$$

Consequently K^* can be equally realized as a generalized Beurling-Ahlfors singular integral transform.

Singular integral on Bergman type space

Define $\mathfrak{B}(\Omega) = \{\nabla u \in L^2(\Omega) \, : \, \Delta u = 0\}$ and

$$\Pi_\Omega(\nabla u)(x) = \mathrm{p.\,v.}\, \nabla_x \int_\Omega \nabla_y G(x,y) \cdot \nabla_y u \, dy, \quad x \in \Omega,$$

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain with connected boundary. The angle operator $P_i(P_d - P_s) : \mathfrak{H}_i \to \mathfrak{H}_i$ is unitarily equivalent to the operator $B_\Omega : \mathfrak{B}(\Omega) \to \mathfrak{B}(\Omega)$,

$$B_\Omega = I + 2\Pi_\Omega.$$

In addition, the operator $K^* : H_0^{-1/2}(\partial\Omega) \to H_0^{-1/2}(\partial\Omega)$ is similar to B_Ω.

Real spectrum

Corollary

The spectrum of $K : H^{1/2} \to H^{1/2}$ *coincides with that of* B_Ω*, except for the point 1. The essential spectra also coincide,*

$$\sigma_{ess}(K) = \sigma_{ess}(B_\Omega).$$

Furthermore, $\sigma(K) \subset \mathbb{R}$ *and any point* $\lambda \in \sigma(K) \setminus \sigma_{ess}(K)$ *is an eigenvalue of finite multiplicity, since* B_Ω *is self-adjoint.*

2D, à la Schiffer

Complex variables help:

$$\nabla_\zeta \nabla_z G(\zeta, z) = \frac{1}{\pi} \frac{1}{(\bar{\zeta} - \bar{z})^2}.$$

Theorem

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded Lipschitz domain with connected boundary and let $T_\Omega : L^2_a(\Omega) \to \overline{L^2_a(\Omega)}$ denote the operator

$$T_\Omega f(z) = \mathrm{p.v.}\, \frac{1}{\pi} \int_\Omega \frac{f(\zeta)}{(\bar{\zeta} - \bar{z})^2}\, dA(\zeta), \quad f \in L^2_a(\Omega),\ z \in \Omega.$$

Then $K^* : H_0^{-1/2}(\partial\Omega) \to H_0^{-1/2}(\partial\Omega)$ is similar to $\overline{T_\Omega} : L^2_a(\Omega) \to L^2_a(\Omega)$, when the spaces are considered over the field of reals. Here $\overline{T_\Omega} f(z) = \overline{T_\Omega f(z)}$.

Symmetric spectrum

Note that the operator T_Ω is defined regardless of topological assumptions on Ω such as boundedness and smoothness. It is also straightforward to check that if L is a fractional linear transformation, then T_Ω and $T_{L(\Omega)}$ are unitarily equivalent.

Corollary

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded Lipschitz domain with connected boundary. Excepting the point 1, the spectrum $\sigma(K)$ of $K : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is symmetric with respect to the origin.

Corollary

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded Lipschitz domain with connected boundary. Then $\sigma(\overline{T_\Omega}) = \sigma(\overline{T_{\Omega_e}})$.

Fredholm eigenvalues of a planar domain

For $\partial\Omega$ smooth $\sigma\left(K^*|_{H_0^{-1/2}}\right) = \sigma(\overline{T_\Omega})$ is classically known as the set of Fredholm eigenvalues of Ω.

We can define the largest Fredholm eigenvalue as $|\sigma(\overline{T_\Omega})|$ for any simply connected domain Ω whose boundary is given by a closed Lipschitz curve in $C^* = \mathbb{C} \cup \{\infty\}$.

While another generalized definition is given by Poincaré's variational problem.

Quasiconformal mapping estimates

Theorem (Krushkal)

Let $\Omega \subsetneq \mathbb{R}^2$ *be an unbounded convex domain with piecewise* $C^{1,\alpha}$*-smooth boundary,* $\alpha > 0$*. Denote by* $0 < \theta < \pi$ *the least interior angle made between the boundary arcs of* $\partial\Omega$*, taking into consideration also the angle made at* ∞*. Then the largest Fredholm eigenvalue of* Ω *is* $1 - \theta/\pi$*.*

Approximation

Working with T_Ω is more flexible than computing singular integrals on the boundary, for instance in approximations of the type $\Omega_n \to \Omega$ corresponding to SOT convergence of $T_{\Omega_n} \to T_\Omega$. New proof of:

Theorem (Kühnau)

Let $\Omega \subset \mathbb{R}^2$ be a $C^{1,\alpha}$-smooth curvilinear polygon. Then

$$|\sigma(T_\Omega)| \geq \max_{1\leq j\leq N} |1 - \theta_j/\pi|$$

Conformal mapping application

Theorem

Let Ω be a $C^{1,\alpha}$-smooth curvilinear polygon with $0 < \theta_j < \pi$ for $1 \leq j \leq N$ such that its angles satisfy

$$\sum_{j=1}^{N-1} (\pi - \theta_j) + \pi + \theta_N \leq 2\pi,$$

possibly after a cyclic permutation of the vertex labels. Then

$$|\sigma_{ess}(K)| = |\sigma_{ess}(T_\Omega)| \leq \max_{1 \leq j \leq N} (1 - \theta_j/\pi),$$

where the double layer potential K is considered as an operator on $H^{1/2}(\partial\Omega)$.

Spectrum of the NP operator in other functional spaces

Theorem (Radon, 1919)

Assume $\partial\Omega$ has bounded rotation. Then

$$|\sigma_{ess}(K)| = \max\left(1 - \theta/\pi\right).$$

Proved by Carleman in his PhD dissertation, in 1916.

Extended to 3D and (some) Hölder space norms by Kral, Netuka, Maz'ya and collaborators.

Lebesgue spaces

Shelepov's formula in L^2:

$$|\sigma_{\mathrm{ess}}(K)| = \max\left(1 - \sin\frac{\theta}{2}\right).$$

Irina Mitrea (2002): $L^p(\partial\Omega), 1 < p < \infty$, with finitely many corners on $\partial\Omega$ produces a non-real spectrum of K, with a closed leminscate for each vertex.

Particular cases and numerical experiments

Werner 1997: $\sigma(T_R)$ of a rectangle R depends on the proportion of the sides.

Helsing and Perfekt 2012: *On the polarizability and capacitance of the cube*

Surveys

M. Costabel: Some historical remarks on the positivity of boundary integral operators. In: Boundary Element Analysis (M. Schanz, O. Steinbach eds.). Lect. Notes Appl. Comp. Mechanics 29, Springer Berlin (2007) 128.

V.G. Mazya: Boundary integral equations. In: Encyclopaedia of Mathematical Sciences 27, Analysis IV (V.G. Mazya, S.M. Nikolski eds.) Springer-Verlag Berlin (1991) 127 - 222.

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