Riemannian and subriemannian geometries on the Heisenberg group and operators generated by their combination

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The Heisenberg group

The CR-structure on the boundary

$$\partial D_{n+1} = \left\{(z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} \,|\, z_{n+1} = t + i\frac{|z|^2}{4}\right\}$$

of the Siegel domain biholomorphically equivalent to the unit ball of $\mathbb{C}^{n+1}$ is defined by the tangent subbundle $T^{1,0}(\partial D_{n+1})$ spanned by the vector fields

$$Z_j = \partial_{z_j} - i\,\frac{\bar{z}_j}{4}\,\partial_t\ .$$

The real and imaginary parts of Z_j and the derivative $T = \partial_t$ span the Heisenberg Lie algebra $\mathfrak{h}_n$. From the Lie algebra one obtains a Lie group structure on $\mathbb{C}^n \times \mathbb{R}$, called the *Heisenberg group* and denoted by H_n.

Under the identification $(z, t) \longmapsto (z, t + i|z|^2/4)$ of H_n with ∂D_{n+1}, left translations on H_n become traces of biholomorphisms of D_{n+1}.

The fact that ∂D_{n+1} is rotationally invariant in z is reflected by the fact that unitary transformations in the z-space are automorphisms of H_n.

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U_n-invariant metrics on H_n

Up to scalings, there is a unique left-invariant and U_n-invariant Riemannian metric on H_n, in which

$$\sqrt{2}Z_1\,, \quad \dots \quad , \sqrt{2}Z_n\,, \sqrt{2}\overline{Z}_1\,, \quad \dots \quad , \sqrt{2}\overline{Z}_n\,, T$$

is an orthonormal basis at each point. The corresponding Laplace-Beltrami operator is

$$\Delta_0 = -2\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_jZ_j) - T^2\,.$$

Besides this, H_n admits a left-invariant and U_n-invariant subriemannian metric, given by restricting the riemannian metric on H_n to its complex tangent subbundle. The corresponding sublaplacian is

$$L = -2\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_jZ_j)\,.$$

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Distance from the origin

The subriemannian distance is bi-Lipschitz-equivalent to the Korányi distance:

$$d_{\mathrm{sub}}((z,t),(0,0)) \sim \left(\frac{|z|^4}{16} + t^2\right)^{1/4}.$$

The riemannian distance is locally equivalent to the euclidean distance, and equivalent to the subriemannian distance at infinity:

$$d_{\mathrm{riem}}((z,t),(0,0)) \sim \begin{cases} (|z|^2 + t^2)^{1/2} & \text{if } \ll 1 \\ \left(\frac{|z|^4}{16} + t^2\right)^{1/4} & \text{if } \gg 1. \end{cases}$$

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U_n-invariant operators

Every (differential, or bounded, or self-adjoint) operator which is left-invariant and U_n-invariant is a function of the sublaplacian L and of iT, where $T = \partial_t$.

This is the case for

- the Laplace-Beltrami operator $\Delta_0 = L - T^2$;
- the Kohn Laplacians $L - i(n - 2q)T$ on $(0, q)$-forms;
- the Folland-Stein operators $L + i\alpha T$;
- the Cauchy-Szegö projection on ∂D_{n+1},

$$Cf = f * \left(\frac{|z|^2}{4} + it\right)^{-n-1} ;$$

- the Riesz transforms, riemannian and subriemannian:

$$Z_j L^{-\frac{1}{2}} , \quad L^{-\frac{1}{2}} Z_j , \quad Z_j \Delta_0^{-\frac{1}{2}} , \quad T\Delta_0^{-\frac{1}{2}} , \quad \text{etc.}$$

Cauchy-Szegö projections and Riesz transforms are singular integral operators (principal value in terms of one of the two metrics).

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The two metrics together

In 1992, D-C. Chang, A. Nagel and E.M. Stein proved that the $\bar{\partial}$-Neumann operator on the boundary of a finite-type weakly pseudoconvex domain domain in $\mathbb{C}^2$ is given by operators which mix together the riemannian metric induced from the ambient space and the subriemannian metric on the boundary complex tangent bundle.

Earlier, D.H. Phong and E.M. Stein had studied singular integral operators on $\mathbb{R}^n$ and on H_n, whose kernels are products of two components with different homogeneities (isotropic and parabolic) and determined to a large extent their boundedness properties on L^p-spaces.

In 1995, D. Müller, F. R. and E.M. Stein studied general multiplier operators $m(L, iT)$, completing, among other things, the questions left open by Phong and Stein.

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Differential forms on H_n

D. Müller, M.M. Peloso, F. R., *Analysis of the Hodge Laplacian on the Heisenberg group*, to appear in Memoirs AMS

Orthonormal basis:

$$\frac{1}{\sqrt{2}} dz_j\,, \qquad \frac{1}{\sqrt{2}} d\bar{z}_j\,, \qquad \theta = dt + \frac{i}{4}\sum_{j=1}^{n} (\bar{z}_j\, dz_j - z_j\, d\bar{z}_j)\,.$$

Differential:

$$df = \underbrace{\sum_{j=1}^{n} Z_j f\, dz_j}_{\partial f} + \underbrace{\sum_{j=1}^{n} \bar{Z}_j f\, d\bar{z}_j}_{\bar{\partial} f} + Tf\,\theta\,.$$

Horizontal (p,q)-forms: $\sum_{|\alpha|=p\,,\,|\beta|=q} f_{\alpha,\beta}(z,t)\, dz^\alpha d\bar{z}^\beta$.

Horizontal differential: $d_H f = \partial f + \bar{\partial} f$

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The Hodge-de Rham Laplacian

Laplacians on horizontal forms:

$$\Box = \partial\partial^* + \partial^*\partial \,, \qquad \overline{\Box} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

$$\Delta_H = d_H d_H^* + d_H^* d_H = \Box + \overline{\Box} \,.$$

On (p,q)-forms (componentwise)

$$\Box = \frac{1}{2}(L + i(n-2p)T) \,, \qquad \overline{\Box} = \frac{1}{2}(L - i(n-2q)T)$$

For the Hodge Laplacian $\Delta_k = dd^* + d^*d$ on k-forms, set

$$\omega = \omega_1 + \theta \wedge \omega_2$$

with ω_1, ω_2 horizontal. Then

$$\Delta_k \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \Delta_H - T^2 + e(d\theta)e(d\theta)^* & i\bar\partial - i\partial \\ i\partial^* - i\bar\partial^* & \Delta_H - T^2 + e(d\theta)^* e(d\theta) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

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Functional calculus on (sub)-laplacians

With A denoting any of the above defined (sub)-laplacians, one wants to understand the L^p-boundedness properties of

- multiplier operators $m(A)$ with m bounded on $\mathbb{R}^+$;
- the appropriate Riesz transforms.

For operators acting on scalar-valued functions, these problems have been extensively studied in different contexts (Lie groups, riemannian manifolds, etc.). One has L^p-boundedness of $1 < p < \infty$.

Notice that $\square$, $\overline{\square}$, Δ_H are "scalar" operators: each of them acts separately on each component $f_{\alpha,\beta}\, dz^\alpha\, d\bar{z}^\beta$ of the form.

The situation is completely different for Δ_k when $0 < k < 2n + 1$.

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The situation is completely different for Δ_k when $0 < k < 2n+1$.

Analysis of Δ_k

Denote by $L^2\Lambda^k$ the space of square-integrable k-forms on H_n.

- L^2 part: decompose $L^2\Lambda^k$ into finitely many closed subspaces V_j, such that, for each j,
  - (i) $\Delta_k : V_j \longrightarrow V_j$;
  - (ii) $\Delta_{k|_{V_j}}$ can be reduced to a scalar operator D_j, possibly up to an explicit unitary transformation of V_j into a different space of forms.
- L^p part ($1 < p < \infty$): prove that
  - (iii) the decomposition into the V_j also takes place in L^p, i.e, the orthogonal projections $\pi_j : L^2\Lambda^k \longrightarrow V_j$ are L^p-bounded;
  - (iv) when a unitary transformation is required, this is also L^p-bounded.

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Analysis of Δ_k

Denote by $L^2\Lambda^k$ the space of square-integrable k-forms on H_n.

- L^2 part: decompose $L^2\Lambda^k$ into finitely many closed subspaces V_j, such that, for each j,
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Examples of V_j, D_j, π_j

The first example is the space $V_1 \subset L^2\Lambda^k$ consisting of the horizontal $(0,k)$ forms ω with $\bar\partial^*\omega = 0$.

$$\Delta_k \omega = \Delta_k \begin{pmatrix} \omega \\ \\ 0 \end{pmatrix} = \begin{pmatrix} (\Delta_H - T^2 + e(d\theta)e(d\theta)^*)\omega \\ \\ (i\partial^* - i\bar\partial^*)\omega \end{pmatrix} = (L - ikT - T^2)\omega \ .$$

The projection π_1, applied to a general $\omega \in L^2\Lambda^k$, first extracts its horizontal $(0,k)$-component, and then projects it onto the subspace of $\bar\partial^*$-closed forms. This second operation is the second-order subriemannian Riesz transform

$$\bar\partial^* \overline{\Box}^{-1} \bar\partial = (\bar\partial^* \overline{\Box}^{-\frac{1}{2}})(\overline{\Box}^{-\frac{1}{2}} \bar\partial) = \bar r_k^* \bar r_k \ .$$

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The second example consists, loosely speaking, of the differentials of forms in the previous example.

To be precise, we introduce the riemannian Riesz transforms

$$R_k = d\Delta_k^{-\frac{1}{2}} : L^2\Lambda^k \longrightarrow L^2\Lambda^{k+1} , \qquad R_k^* = \Delta_k^{-\frac{1}{2}} d^* : L^2\Lambda^{k+1} \longrightarrow L^2\Lambda^k$$

and set $V_2 = R_k V_1$. With $\omega = R_k\sigma$, we have

$$\Delta_{k+1}\omega = \Delta_{k+1} R_k\sigma = R_k(\Delta_k\sigma) = R_k(L - ikT - T^2)\sigma .$$

In this case, reduction to the scalar operator $D_2 = L - ikT - T^2$ is obtained at the cost of replacing forms in V_2 by forms in V_1 via the unitary transformation R_k.

The chain of operations to be applied to project a general k-form into V_2 is, for $k < n$,

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General properties

The general features of the operators arising in the decompositon of Δ_k turn out to be the following:

- the operators D_j have the form $A_j \pm \sqrt{B_j}$, where A_j and B_j are linear combinations L, T, T^2 plus a constant term;
- the orthogonal projections π_j are expressed in terms of compositions of operators of the following kinds:
  - (i) the riemannian Riesz transforms R_ℓ, R_ℓ^* of full forms, with $\ell < k$,
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Nature of the operators involved

Some of the above operators are given by convolution with singular integral (or Calderón-Zygmund) kernels, $Af = f * K$, of some kind.

The kind depends on the given metric. We distinguish between K_{sub} and K_{riem}.

Properties of K_{sub}:

$$\left| Z^{\alpha} \bar{Z}^{\beta} T^{\gamma} K_{\text{sub}}(x) \right| \lesssim d_{\text{sub}}(x,0)^{-|\alpha|-|\beta|-2|\gamma|} V_{\text{sub}}(x)^{-1} ,$$

where $V_{\text{sub}}(x)$ is the volume of the ball of radius $d(x,0)$.

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Subriemannian singular integrals

The following are singular integral operators of subriemannian type:

- the subriemannian Riesz transforms, r_{pq} etc.;
- multiplier operators $m(L, iT)$, with $m(\xi, \lambda)$ satisfying the inequalities

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Multipliers and kernels

The multipliers m arising in the decomposition of $L^2\Lambda^k$ are neither riemannian or subriemannian. They satisfy a *mixed* condition:

$$\left|\partial_\xi^j \partial_\lambda^k m(\xi,\lambda)\right| \lesssim \big(|\xi|+|\lambda|\big)^{-j}\big(|\xi|+\lambda^2\big)^{-\frac{k}{2}} .$$

The corresponding operators $m(L, iT)$ have kernels K which also satisfy mixed inequalities near 0:

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The proof of this fact is interlaced with the proof of the decomposition of $L^p\Lambda_k$ by an inductive argument:

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Applications to L^p-cohomology of H_n as a riemannian manifold (Strichartz, Li, Auscher-Coulhon-Duong-Hofmann).

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