Riemannian and subriemannian geometries on the Heisenberg group and operators generated by their combination

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The Heisenberg group

The CR-structure on the boundary

\[ \partial D_{n+1} = \left\{ (z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} \mid z_{n+1} = t + i \frac{|z|^2}{4} \right\} \]

of the Siegel domain biholomorphically equivalent to the unit ball of \( \mathbb{C}^{n+1} \) is defined by the tangent subbundle \( T^{1,0}(\partial D_{n+1}) \) spanned by the vector fields

\[ Z_j = \partial z_j - i \frac{\bar{z}_j}{4} \partial t. \]

The real and imaginary parts of \( Z_j \) and the derivative \( T = \partial_t \) span the Heisenberg Lie algebra \( \mathfrak{h}_n \). From the Lie algebra one obtains a Lie group structure on \( \mathbb{C}^n \times \mathbb{R} \), called the Heisenberg group and denoted by \( H_n \).

Under the identification \((z, t) \mapsto (z, t + i|z|^2/4)\) of \( H_n \) with \( \partial D_{n+1} \), left translations on \( H_n \) become traces of biholomorphisms of \( D_{n+1} \).

The fact that \( \partial D_{n+1} \) is rotationally invariant in \( z \) is reflected by the fact that unitary transformations in the \( z \)-space are automorphisms of \( H_n \).
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Under the identification $(z, t) \mapsto (z, t + i|z|^2/4)$ of $H_n$ with $\partial D_{n+1}$, left translations on $H_n$ become traces of biholomorphisms of $D_{n+1}$.

The fact that $\partial D_{n+1}$ is rotationally invariant in $z$ is reflected by the fact that unitary transformations in the $z$-space are automorphisms of $H_n$. 
**$U_n$-invariant metrics on $H_n$**

Up to scalings, there is a unique left-invariant and $U_n$-invariant Riemannian metric on $H_n$, in which

$$\sqrt{2}Z_1, \ldots, \sqrt{2}Z_n, \sqrt{2}\bar{Z}_1, \ldots, \sqrt{2}\bar{Z}_n, T$$

is an orthonormal basis at each point. The corresponding Laplace-Beltrami operator is

$$\Delta_0 = -2\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j) - T^2.$$

Besides this, $H_n$ admits a left-invariant and $U_n$-invariant subriemannian metric, given by restricting the riemannian metric on $H_n$ to its complex tangent subbundle. The corresponding sublaplacian is

$$L = -2\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j).$$
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Distance from the origin

The subriemannian distance is bi-Lipschitz-equivalent to the Korányi distance:

\[ d_{\text{sub}}((z, t), (0, 0)) \sim \left( \frac{|z|^4}{16} + t^2 \right)^{1/4}. \]

The riemannian distance is locally equivalent to the euclidean distance, and equivalent to the subriemannian distance at infinity:

\[ d_{\text{riem}}((z, t), (0, 0)) \sim \begin{cases} 
  (|z|^2 + t^2)^{1/2} & \text{if } \ll 1 \\
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Every (differential, or bounded, or self-adjoint) operator which is left-invariant and $U_n$-invariant is a function of the sublaplacian $L$ and of $iT$, where $T = ∂_t$.

This is the case for

- the Laplace-Beltrami operator $Δ_0 = L - T^2$;
- the Kohn Laplacians $L - i(n - 2q)T$ on $(0, q)$-forms;
- the Folland-Stein operators $L + iαT$;
- the Cauchy-Szegő projection on $∂D_{n+1}$,
  \[ Cf = f ∗ \left( \frac{|Z|^2}{4} + it \right)^{-n-1}; \]
- the Riesz transforms, riemannian and subriemannian:
  \[ Z_jL^{-\frac{1}{2}}, \quad L^{-\frac{1}{2}}Z_j, \quad Z_jΔ_0^{-\frac{1}{2}}, \quad TΔ_0^{-\frac{1}{2}}, \quad \text{etc.} \]

Cauchy-Szegő projections and Riesz transforms are singular integral operators (principal value in terms of one of the two metrics).
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In 1992, D-C. Chang, A. Nagel and E.M. Stein proved that the $\bar{\partial}$-Neumann operator on the boundary of a finite-type weakly pseudoconvex domain domain in $\mathbb{C}^2$ is given by operators which mix together the riemannian metric induced from the ambient space and the subriemannian metric on the boundary complex tangent bundle.

Earlier, D.H. Phong and E.M. Stein had studied singular integral operators on $\mathbb{R}^n$ and on $\mathcal{H}_n$, whose kernels are products of two components with different homogeneities (isotropic and parabolic) and determined to a large extent their boundedness properties on $L^p$-spaces.

In 1995, D. Müller, F. R. and E.M. Stein studied general multiplier operators $m(L, iT)$, completing, among other things, the questions left open by Phong and Stein.

Extensions to more general situations have been recently studied by A. Nagel, F. R., E.M. Stein, S. Wainger and, independently, by B. Street.
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Differential forms on $H_n$


Orthonormal basis:

\[ \frac{1}{\sqrt{2}} dz_j , \quad \frac{1}{\sqrt{2}} d\bar{z}_j , \quad \theta = dt + \frac{i}{4} \sum_{j=1}^{n} (\bar{z}_j dz_j - z_j d\bar{z}_j) . \]

Differential:

\[ df = \sum_{j=1}^{n} Z_j f \ dz_j + \sum_{j=1}^{n} \bar{Z}_j f \ d\bar{z}_j + T f \ \theta . \]

Horizontal differential:

\[ d_H f = \partial f + \bar{\partial} f \]

Horizontal $(p, q)$-forms: \[ \sum_{|\alpha|=p, \ |\beta|=q} f_{\alpha, \beta}(z, t) \ dz^\alpha \ d\bar{z}^\beta . \]
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The Hodge-de Rham Laplacian

Laplacians on horizontal forms:

\[ \Box = \partial \partial^* + \partial^* \partial, \quad \bar{\Box} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} \]

\[ \Delta_H = d_H d_H^* + d_H^* d_H = \Box + \bar{\Box}. \]

On \((p, q)\)-forms (componentwise)

\[ \Box = \frac{1}{2} (L + i(n - 2p) T), \quad \bar{\Box} = \frac{1}{2} (L - i(n - 2q) T) \]

For the Hodge Laplacian \(\Delta_k = dd^* + d^* d\) on \(k\)-forms, set

\[ \omega = \omega_1 + \theta \wedge \omega_2 \]

with \(\omega_1, \omega_2\) horizontal. Then

\[
\Delta_k \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \Delta_H - T^2 + e(d\theta)e(d\theta)^* & i\bar{\partial} - i\partial \\ i\partial^* - i\bar{\partial}^* & \Delta_H - T^2 + e(d\theta)^*e(d\theta) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}
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With $A$ denoting any of the above defined (sub)-laplacians, one wants to understand the $L^p$-boundedness properties of

- multiplier operators $m(A)$ with $m$ bounded on $\mathbb{R}^+$;
- the appropriate Riesz transforms.

For operators acting on scalar-valued functions, these problems have been extensively studied in different contexts (Lie groups, riemannian manifolds, etc.). One has $L^p$-boundedness of $1 < p < \infty$.

Notice that $\Box$, $\bar{\Box}$, $\Delta_H$ are “scalar” operators: each of them acts separately on each component $f_{\alpha, \beta} \, dz^{\alpha} \, d\bar{z}^{\beta}$ of the form.

The situation is completely different for $\Delta_k$ when $0 < k < 2n + 1$. 

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Analysis of $\Delta_k$

Denote by $L^2 \Lambda^k$ the space of square-integrable $k$-forms on $H_n$.

- **$L^2$ part**: decompose $L^2 \Lambda^k$ into finitely many closed subspaces $V_j$, such that, for each $j$,
  
  (i) $\Delta_k : V_j \rightarrow V_j$;
  
  (ii) $\Delta_k|_{V_j}$ can be reduced to a scalar operator $D_j$, possibly up to an explicit unitary transformation of $V_j$ into a different space of forms.

- **$L^p$ part ($1 < p < \infty$)**: prove that
  
  (iii) the decomposition into the $V_j$ also takes place in $L^p$, i.e., the orthogonal projections $\pi_j : L^2 \Lambda^k \rightarrow V_j$ are $L^p$-bounded;
  
  (iv) when a unitary transformation is required, this is also $L^p$-bounded.
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Denote by $L^2 \Lambda^k$ the space of square-integrable $k$-forms on $H_n$.

- **$L^2$ part**: decompose $L^2 \Lambda^k$ into finitely many closed subspaces $V_j$, such that, for each $j$,
  1. $\Delta_k : V_j \rightarrow V_j$;
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- **$L^p$ part ($1 < p < \infty$)**: prove that
  3. the decomposition into the $V_j$ also takes place in $L^p$, i.e., the orthogonal projections $\pi_j : L^2 \Lambda^k \rightarrow V_j$ are $L^p$-bounded;
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Examples of $V_j$, $D_j$, $\pi_j$

The first example is the space $V_1 \subset L^2 \Lambda^k$ consisting of the horizontal $(0, k)$ forms $\omega$ with $\bar{\partial}^* \omega = 0$.

$$\Delta_k \omega = \Delta_k \begin{pmatrix} \omega \\ 0 \end{pmatrix} = \begin{pmatrix} (\Delta_H - T^2 + e(d\theta)e(d\theta)^*)\omega \\ (i\partial^* - i\bar{\partial}^*)\omega \end{pmatrix} = (L - ikT - T^2)\omega .$$

The projection $\pi_1$, applied to a general $\omega \in L^2 \Lambda^k$, first extracts its horizontal $(0, k)$-component, and then projects it onto the subspace of $\bar{\partial}^*$-closed forms. This second operation is the second-order subriemannian Riesz transform

$$\bar{\partial}^* \square^{-1} \bar{\partial} = (\bar{\partial}^* \square^{-\frac{1}{2}}) (\square^{-\frac{1}{2}} \bar{\partial}) = \bar{r}_k^* r_k .$$

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The second example consists, loosely speaking, of the differentials of forms in the previous example.

To be precise, we introduce the riemannian Riesz transforms

$$R_k = d\Delta_k^{-\frac{1}{2}} : L^2 \Lambda^k \longrightarrow L^2 \Lambda^{k+1}, \quad R_k^* = \Delta_k^{-\frac{1}{2}} d^* : L^2 \Lambda^{k+1} \longrightarrow L^2 \Lambda^k$$

and set $V_2 = R_k V_1$. With $\omega = R_k \sigma$, we have

$$\Delta_{k+1} \omega = \Delta_{k+1} R_k \sigma = R_k (\Delta_k \sigma) = R_k (L - ikT - T^2) \sigma.$$

In this case, reduction to the scalar operator $D_2 = L - ikT - T^2$ is obtained at the cost of replacing forms in $V_2$ by forms in $V_1$ via the unitary transformation $R_k$.

The chain of operations to be applied to project a general $k$-form into $V_2$ is, for $k < n$,

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General properties

The general features of the operators arising in the decomposition of $\Delta_k$ turn out to be the following:

- the operators $D_j$ have the form $A_j \pm \sqrt{B_j}$, where $A_j$ and $B_j$ are linear combinations $L, T, T^2$ plus a constant term;
- the orthogonal projections $\pi_j$ are expressed in terms of compositions of operators of the following kinds:
  1. Riemannian Riesz transforms $R_\ell, R_\ell^*$ of full forms, with $\ell < k$,
  2. Subriemannian Riesz transforms $r_{pq}, r_{pq}^*, \tilde{r}_{pq}, \tilde{r}_{pq}^*$ of horizontal forms, with $p + q < k$,
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Nature of the operators involved

Some of the above operators are given by convolution with singular integral (or Calderón-Zygmund) kernels, $Af = f \ast K$, of some kind. The kind depends on the given metric. We distinguish between $K_{\text{sub}}$ and $K_{\text{riem}}$.

Properties of $K_{\text{sub}}$:

$$|Z^\alpha Z^\beta T^\gamma K_{\text{sub}}(x)| \lesssim d_{\text{sub}}(x, 0)^{-|\alpha|-|\beta|-2|\gamma|} V_{\text{sub}}(x)^{-1},$$

where $V_{\text{sub}}(x)$ is the volume of the ball of radius $d(x, 0)$.

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Subriemannian singular integrals

The following are singular integral operators of subriemannian type:

- the subriemannian Riesz transforms, $r_{pq}$ etc.;
- multiplier operators $m(L, iT)$, with $m(\xi, \lambda)$ satisfying the inequalities

$$\left| \partial_{\xi}^j \partial_{\lambda}^k m(\xi, \lambda) \right| \lesssim (|\xi| + |\lambda|)^{-j-k}.$$  

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All these operators are scalar, and are well known to be $L^p$-bounded, for $1 < p < \infty$. 
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Multipliers and kernels

The multipliers $m$ arising in the decomposition of $L^2 \Lambda^k$ are neither riemannian or subriemannian. They satisfy a *mixed* condition:

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The corresponding operators $m(L, iT)$ have kernels $K$ which also satisfy mixed inequalities near 0:

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and behave like subriemannian kernels at infinity.

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The riemannian Riesz transforms

Before concluding that all ingredients of the decomposition of $L^p \Lambda_k$ (projections and intertwining operators), the only missing item is that the riemannian Riesz transform $R_k = d \Delta_k^{-\frac{1}{2}}$ is $L^p$-bounded for $1 < p < \infty$.

The proof of this fact is interlaced with the proof of the decomposition of $L^p \Lambda_k$ by an inductive argument:

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\text{decomposition of } L^p \Lambda_k, \quad \Rightarrow \quad R_k \text{ is } L^p\text{-bounded for } 1 < p < \infty \\
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Applications to $L^p$-cohomology of $H_n$ as a riemannian manifold (Strichartz, Li, Auscher-Coulhon-Duong-Hofmann).
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The proof of this fact is interlaced with the proof of the decomposition of $L^p \Lambda_k$ by an inductive argument:

$$\text{decomposition of } L^p \Lambda_k \implies R_k \text{ is } L^p\text{-bounded for } 1 < p < \infty \implies \text{decomposition of } L^p \Lambda_{k+1}$$

Applications to $L^p$-cohomology of $H_n$ as a riemannian manifold (Strichartz, Li, Auscher-Coulhon-Duong-Hofmann).