Analytic Structure in Spectra of Multiplier Algebras

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Hilbert Function Spaces
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Overview

- I am going to talk about structure in the maximal ideal space of the multiplier algebras of certain Dirichlet type spaces.
- The main conclusions are:
    - For each of the Dirichlet type spaces $\mathcal{D}_\alpha$, $0 < \alpha < 1$, there is analytic structure in the spectrum of its multiplier algebra. This is similar to the situation for the model case; $\mathcal{D}_0 = H^2$ and its multiplier algebra H^∞.
    - For the Dirichlet space, $\mathcal{D}_1$, there is structure, but the details of the structure are not clear.

Outline

1. Introduction of the players.
2. Summary of classical results.
3. Analysis for the spaces $\mathcal{D}_\alpha$, $0 < \alpha < 1$.
    1. Showing there is analytic structure.
    2. How far does the analogy go?
4. Analysis on the Dirichlet space, $\mathcal{D}_1$,
    1. Why is the situation fundamentally different?
    2. What we know.
5. Final Comments.

Introduction of the Players

- For $0 \leq \alpha \leq 1$, $\mathcal{D}_\alpha$ is the Hilbert space of $f \in \operatorname{Hol}(\mathbb{D})$.
  $f = \sum a_n z^n \in \mathcal{D}_\alpha$ such that:

$$\|f\|_\alpha^2 = \sum (n+1)^\alpha |a_n|^2 \approx \|f\|_{H^2}^2 + \int\int_{\mathbb{D}} |f'|^2 (1-|z|^2)^{1-\alpha} dA < \infty.$$

- $\mathcal{D}_0$ is the Hardy space; $\mathcal{D}_1$ is the Dirichlet space.
- The multiplier algebras: $\mathfrak{M}_\alpha = \{m : f \to mf \text{ bounded on } \mathcal{D}_\alpha\}$. With the operator norm $\mathfrak{M}_\alpha$ is a commutative Banach algebra.
- Generalized pseudohyperbolic metric: Let $\{\hat{k}_z\}$ be the normalized reproducing kernels for $\mathcal{D}_\alpha$ and set.

$$\delta_\alpha(z,w) = \sqrt{1 - \left|\left\langle \hat{k}_z, \hat{k}_w \right\rangle\right|^2}$$

  δ_0 is the classical pseudohyperbolic metric. Always, $0 \leq \delta_\alpha \leq 1$. There is a Schwarz-Pick style lemma; If $m \in \mathfrak{M}_\alpha$, $\|m\| = 1$, then $\delta_0(m(z), m(w)) \leq \delta_\alpha(z,w)$.

More Players

- $\mathfrak{X}_\alpha$ = spectrum of $\mathfrak{M}_\alpha$ = topologized maximal ideal space = topologized set of nonzero multiplicative linear functionals = generalized bounded point evaluations for $\mathfrak{M}_\alpha$ = a *large* compactification of $\mathbb{D}$ to which every $m \in \mathfrak{M}_\alpha$ extends as a continuous function, $\hat{m}$, its Gelfand transform.
- The map $m \to \hat{m}$ is a contraction of $\mathfrak{M}_\alpha$ into $C(\mathfrak{X}_\alpha)$. It is an isometry if and only if $\mathfrak{M}_\alpha$ is a *uniform algebra*, $\forall m \; \left\| m^2 \right\| = \left\| m \right\|^2$. In our situation this only happens for $\alpha = 0$. Surprisingly, that hasn't influenced the analysis.
- The model case is: $\mathcal{D}_0 = H^2$, $\delta_0 = \rho$, $\mathfrak{M}_0 = H^\infty$, contractive property of δ_0 = Schwarz-Pick lemma.

Gleason's Insight and the Search for Analytic Structure

- The Gleason metric on $\mathfrak{X}_\alpha$ is given by

$$\operatorname{dist}(\mu,\nu) = \sup\left\{ |\hat{m}(\mu)| : \hat{m}(\nu) = 0, \|\hat{m}\|_{\mathfrak{M}_\alpha} = 1 \right\}.$$

- On $\mathbb{D} \subset \mathfrak{X}_\alpha$ this agrees with δ_α and we continue that notation.
- δ_a is a metric on $\mathfrak{X}_\alpha$. $0 \leq \delta_\alpha \leq 1$. $\delta_\alpha(\mu,\nu) < 1$ is an equivalence relation. The equivalence classes are called (Gleason) parts.
- For $m \in \mathfrak{X}_\alpha$, let $\mathcal{P}(m)$ be the part containing m.
- An analytic disk in $\mathfrak{X}_\alpha$ is a nonconstant map $\Phi : \mathbb{D} \to \mathfrak{X}_\alpha$ s.t. $\forall m \in \mathfrak{M}_\alpha$, $\hat{m}(\Phi(z))$ is holomorphic.

Theorem (Gleason 1957)

Any analytic disk is contained in a single part.

Classical Results

- H^∞ is the algebra of all bounded analytic functions on $\mathbb{D}$ with the supremum norm. It equals $\mathfrak{M}_0$, the multiplier algebra of the classical Hardy space.
- A part $\mathcal{P}$ is called an *analytic disk* if the map Φ can be chosen to be a one to one map onto $\mathcal{P}$.
- A set $Z = \{z_i\} \subset \mathbb{D}$ is an interpolating sequence for $\mathfrak{M}_\alpha$ if the restrictions to Z of the functions in $\mathfrak{M}_\alpha$ give all of $\ell^\infty(Z)$.

Theorem (I. J. Schark 1961)

Any point in the closure of an interpolating sequence is in an analytic disk.

Theorem (K. Hoffman 1967)

1. $\forall m \in \mathfrak{M}_0$, *either* $\mathcal{P}(m) = m$ *or* $\mathcal{P}(m)$ *is an analytic disk.*
2. *The second case happens if and only* m *is in the closure of an interpolating sequence.*

Blaschke Products in Multiplier Algebras

Theorem

$0 < \alpha < 1$. *If* $Z = \{z_i\} \subset \mathbb{D}$ *is an interpolating sequence for* $\mathfrak{M}_\alpha$ *then the Blaschke product* B *with zeros at* Z *is in* $\mathfrak{M}_\alpha$. Z *is also an interpolating sequence for* $\mathfrak{M}_0$ *and hence, among other things.*

$$\inf \left|(1 - |z_n|^2)B'(z_n)\right| = \delta > 0. \qquad \text{(Later)}$$

- This is automatic for $\alpha = 0$. Every such Z is the zero set of a Blaschke product and every Blaschke product is in H^∞.
- Two observations get the proof of the theorem started:
  - If Z is an interpolating sequence then $\sum(1 - |z_i|^2)^{1-\alpha}\delta_{z_i}$ is a Carleson measure for $\mathcal{D}_\alpha$.
  - If $(1 - |z_i|^2)^{1-\alpha} |B'(z)|^2 \, dxdy$ is a Carleson measure then $B \in \mathfrak{M}_\alpha$.
- Finish the proof with a "spreading of Carleson measures" lemma.

Analytic Structure in the Spectra

Theorem (Analytic Discs)

Suppose $0 < \alpha < 1$. If $m \in \mathfrak{M}_\alpha$ is in the closure of an interpolating sequence $Z = \{z_n\}$ then m lies in analytic disk.

Using the interpolating Blaschke products, we can run the clasical proof.

- Let B be the Blaschke product with zeros at Z. Let L_n be the disk automorphism which takes 0 to z_n, and $\{L_\gamma(0)\}$ a net which converges to m. General topology insures that the net of maps $L_\gamma : \mathbb{D} \to \mathbb{D}$ has a limit map $L : \mathbb{D} \to \mathfrak{X}_\alpha$ with $L(0) = m$.
- For any $f \in \mathfrak{M}_\alpha$, $\hat{f}(L(z)) = \lim \hat{f}(L_\gamma(z))$ is a limit of bounded holomorphic functions, hence the limit function is holomorphic, and we also have pointwise convergence of the derivatives.
- We now show $(\hat{B} \circ L)'(0) \neq 0$ and conclude L is nonconstant:

$$\begin{aligned}(\hat{B} \circ L)'(0) &= \lim \left(B \circ L_\gamma\right)'(0) = \lim B'(z_\gamma) L'_\gamma(0) \\ &= \lim (1 - |z_\gamma|^2) L'_\gamma(0) \neq 0.\end{aligned}$$

How Far Does This Go?

- Much of Hoffman's analysis is based on the fact that interpolating sequences are zero sets of Blaschke products with very specialzed properties; together with a careful analysis of those products.
- For example, recalling $\delta(B) = \inf \left|(1 - |z_n|^2) B'(z_n)\right|$. B can be factored $B = B_1 B_2$ with $\delta(B_i) \geq \delta(B)^{1/2}$.
- Note that (Later) shows up.
- His results hold for the Blaschke products in $\mathfrak{M}_\alpha$ and hence substantial parts of his analysis go through.
- At other places the path is not clear. For instance: if $f \in \mathfrak{M}_\alpha$, must the Blaschke product with the same zeros as f also be in $\mathfrak{M}_\alpha$?

Those Ideas Don't Work for the Dirichlet Space

- Neither $\mathcal{D}_1$ nor the multiplier algebra $\mathfrak{M}_1$ contain Blaschke products.
- More fundamentally, schemes similar to the previous one can't work.

Theorem

Suppose $\{L_n\}$ are maps of $\mathbb{D}$ to $\mathbb{D}$ and $|L_n(0)| \to 1$. If L is map of $\mathbb{D}$ into $\mathfrak{X}_1$ obtained as a limit of some $\{L_n\}$ then L is a constant map.

Proof.

Pick $w \in \mathbb{D}$ and set $L_n(0) = \theta_n,\ L_n(w) = \lambda_n$

1. By Schwarz-Pick $\forall n$, $1 > \rho(0, w) \geq \rho(\theta_n, \lambda_n)$.
2. Extended elementary calculation shows that: If $1 > c \geq \rho(\theta_n, \lambda_n)$ and $|\theta_n| \to 1$ then $\delta_1(\theta_n, \lambda_n) \to 0$.
3. $0 = \limsup \delta_1(\theta_n, \lambda_n) \geq \delta_1(\lim \theta_n, \lim \lambda_n)$.

□

Positive Results for the Dirichlet Space

Generating new interpolating sequences from old

Theorem (New Interpolating Sequences)

Suppose $0 < \alpha \leq 1$ *and that* $\{z_n\} = Z \in IS_\alpha$. *There is an* $\varepsilon = \varepsilon(S) > 0$ *such that if* $\{w_n\} = W$ *is another sequence in* $\mathbb{D}$ *with* $\delta_\alpha(z_n, w_n) < \varepsilon$, $n = 1, 2, ...,$ *then* $W \in IS_\alpha$.

Proof.

One needs to show that W is separated in the δ_α metric and that the associated measure satisfies a Carleson measure condition.

- The separation follows from the construction.
- Information about the quantities $\delta_\alpha(w_n, w_m)$ gives information about the entries of the Gramm matrix associated with W.
- The Carleson measure condition is verified by working with that Gramm matrix.

Finding Points in the Part

Theorem (Dirichlet Parts)

If $m \in \mathfrak{X}_1$ is in the closure of an interpolating sequence Z, then $\mathcal{P}(m)$ contains infinitely many points.

Proof.

- First we outline how to find one additional point.
  - Use the previous theorem to construct Z' near Z and such that $Z' \cup Z$ is an interpolating sequence.
  - Use the fact that Z' is near Z to show that $\overline{Z'}$ meets $\mathcal{P}(m)$ at some point m'.
  - Use the fact that $Z' \cup Z$ is an interpolatng sequence to show $m' \neq m$.
- To get the full result, construct Z', Z'', Z''' etc.

$\square$

The Generality of Theorem "Dirichlet Parts".

- The previous theorem used the following facts about the Dirichlet space
  - (*) H is a reproducing kernel Hilbert space whose kernel has the Complete Pick Property.
  - (**) If Z is a sequence of points with Gramm matrix $\left(k_{ij}\right)$ and if that matrix is bounded, then so is the matrix $\left(\left|k_{ij}\right|\right)$.
- If we have those then, thanks to many people over the years (Boe especially):
  - The interpolation sequences for both H and its multiplier algebra are characterized by separation in the δ_H metric and the associated measure satisfying a Carleson measure condition.
  - The analog of theorem "New Interpolating Sequences" holds.
- At that point a version of the theorem "Dirichlet Parts" holds.

What About Theorem "Analytic Disks"?

- The proof of the previous theorem also applies to the spaces $\mathcal{D}_\alpha$, $0 < \alpha < 1$. It was not mentioned because the "Analytic Disks" theorem is a much stronger result.
- The proof that was outlined for "Analytic Disks" made fundamental use of Blaschke products. That suggests that the result may not be very general.
- However there is a more general construction.

Generalized Blaschke Products

- Suppose H is a function space which satisfies (*) and (**).
- For $z \in \mathbb{D}$, let β_z be the function in the multiplier algebra which maximizes $\operatorname{Re} \beta_z(0)$, subject to $\|\beta_z\| = 1$, $\beta_z(z) = 0$.
- For $Z = \{z_n\} \subset \mathbb{D}$, let $\mathfrak{B}_Z(w)$ be the associated generalized Blaschke product.
$$\mathfrak{B}_Z(w) = \prod_n \beta_{z_n}(w)$$
(Sawyer, Function theory..., 2009; Ch. 5)
- If Z is an interpolating sequence for H then $\mathfrak{B}_Z$ converges to a multiplier of norm at most one with exact zero set Z..
- For $\mathcal{D}_0 = H^2$ this produces classical Blaschke products. For $\mathcal{D}_\alpha$, $\alpha > 0$ they are different.
- However they could have been used in the earlier proof because
$$\inf \left|(1 - |z_n|^2)\mathfrak{B}'_Z(z_n)\right| = \delta > 0. \tag{1}$$

The Return to "Later"

- We are going to rewrite (1). For functions in the multiplier algebra introduce the generalized differential operator $\mathfrak{D}_H$:

$$\left(\mathfrak{D}_H f\right)(w) = \lim_{\omega \to w} \left| \frac{f(w) - f(\omega)}{\delta_H(w, \omega)} \right|$$

- Suppose now that $H = \mathcal{D}_\alpha$, $\alpha \neq 1$; i.e., not the Dirichlet space. The following are equivalent

1. $$\inf \left|(1 - |z_n|^2)\mathfrak{B}'_Z(z_n)\right| = \delta > 0.$$
2. $$\inf \left|\mathfrak{D}_H \mathfrak{B}_Z(z_n)\right| = \delta > 0.$$

- Although the formulas are fancy, the verifications are elementary and computational.

What About the Dirichlet Space

- Now suppose we consider $H = \mathcal{D}_1$, the Dirichlet space. The definitions continue to make sense. Should we consider
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$$
\begin{aligned}
\inf \left|(1 - |z_n|^2)\mathfrak{B}'_Z(z_n)\right| &= \delta > 0, \text{ or} \\
\inf \left|\mathfrak{D}_H \mathfrak{B}_Z(z_n)\right| &= \delta > 0?
\end{aligned}
$$

- The earlier proof suggests trying to salvage the first.
- However the first always fails and the second is always true.
- Perhaps this suggests a way to go forward.

Thank You !!