Reverse Carleson measures on $\mathscr{H}(b)$ spaces

A. Blandigneres, E. Fricain, F. Gaunard, A. Hartmann, W. Ross

Gargnano 2013

Set up

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- $\mathscr{H}$ is a Hilbert space of analytic functions on $\mathbb{D}$
- $\mu \in M_+(\mathbb{D})$ (finite positive Borel measures on $\mathbb{D}$)
- $$\|f\|_\mu := \left(\int |f|^2 d\mu\right)^{1/2}$$
- μ is a *direct Carleson* measure for $\mathscr{H}$ if
$$\|f\|_\mu \lesssim \|f\|_{\mathscr{H}}, \quad f \in \mathscr{H}$$
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We often want to write

$$\|f\|_\mu \lesssim \|f\|_{\mathscr{H}} \quad \text{and} \quad \|f\|_{\mathscr{H}} \lesssim \|f\|_\mu \quad \text{for} \quad \mu \in M_+(\mathbb{D}^-).$$

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Reproducing kernel thesis

Suppose

$$k_\lambda^{\mathscr{H}}, \quad \lambda \in \mathbb{D},$$

are the reproducing kernels for $\mathscr{H}$, i.e.,

$$f(\lambda) = \langle f, k_\lambda^{\mathscr{H}} \rangle_{\mathscr{H}}, \quad \lambda \in \mathbb{D}, f \in \mathscr{H}.$$

One can often need only test

$$\|k_\lambda^{\mathscr{H}}\|_{\mathscr{H}} \lesssim \|k_\lambda^{\mathscr{H}}\|_{\mu}$$

or

$$\|k_\lambda^{\mathscr{H}}\|_{\mu} \lesssim \|k_\lambda^{\mathscr{H}}\|_{\mathscr{H}}$$

This is called the *reproducing kernel thesis.*

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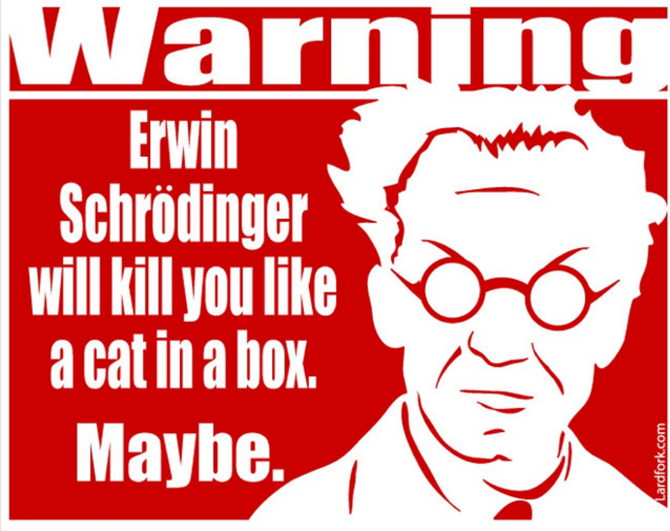
Warning
Erwin
Schrödinger
will kill you like
a cat in a box.
Maybe.
Lardfork.com

Carleson's results for H^2

A classical result

Theorem (Carleson (1962))

For $\mu \in M_+(\mathbb{D}^-)$ TFAE

1. $$\int |f|^2 d\mu \lesssim \int |f|^2 dm, \quad f \in H^2 \cap \mathcal{C}(\mathbb{D}^-);$$

2. $$\int |k_\lambda|^2 d\mu \lesssim \int |k_\lambda|^2 dm, \quad \lambda \in \mathbb{D};$$

3. $$\sup_I \frac{\mu(S(I))}{m(I)} < +\infty.$$

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Generalized like crazy

- Carleson - Hardy space
- Hastings/Oleinik/Luecking - Bergman space
- Wu/Arcozzi/Rochberg/Sawyer - (certain) Dirichlet spaces
- Girela/Palaez - (certain other types of) Dirichlet spaces
- Chacon - (certain other types of) Dirichlet spaces
- Arcozzi/Rochberg/Sawyer - Besov spaces
- Spaces on $\mathbb{B}^n$
- Spaces on pseudo-convex domains in $\mathbb{C}^n$
- Aleksandrov/Volberg/Treil/Baranov - $(\Theta H^2)^\perp$ spaces
- Baranov/Fricain/Mashreghi - $\mathscr{H}(b)$ spaces

Reverse Carleson for H^2

Reverse classical result

Theorem (Queffelec et al (2010), Hartmann et al (2013))

For $\mu \in M_+(\mathbb{D}^-)$ *TFAE*

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Other RCE

- Bergman spaces (Luecking 1985)
- Dirichlet-type spaces (Chacon 2010)
- $(\Theta H^2)^\perp$
  - deBranges
  - Aleksandrov
  - Treil
  - Volberg
  - Baranov
  - BFGHR

The basics of deBranges-Rovnyak spaces

$\mathscr{H}(b)$ spaces

- $\mathbf{b}(H^\infty) = \{g \in H^\infty(\mathbb{D}) : \|g\|_\infty \leqslant 1\}$
- For $b \in \mathbf{b}(H^\infty)$
$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}$$
- For $\lambda_1, \cdots \lambda_n \in \mathbb{D}, c_1, \cdots, c_n \in \mathbb{C}$,
$$\sum_{i,j} c_i \overline{c_j} k_{\lambda_i}^b(\lambda_j) > 0.$$
- Define
$$\left\| \sum_{j=1}^n c_j k_{\lambda_j}^b \right\|_b^2 := \sum_{1 \leqslant i,j \leqslant n} c_i \overline{c_j} k_{\lambda_i}^b(\lambda_j)$$
- $\mathscr{H}(b)$ closure of finite linear combos of kernels.

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$\mathscr{H}(b)$ facts

- k_λ^b is the reproducing kernel for $\mathscr{H}(b)$, i.e.,

$$f(\lambda) := \langle f, k_\lambda^b \rangle_b, \quad \lambda \in \mathbb{D}, f \in \mathscr{H}(b).$$

- $\mathscr{H}(b)$ is contractively contained in H^2.
- When $\|b\|_\infty < 1$, then $\mathscr{H}(b) = H^2$ with an equivalent norm.
- When b is inner, then $\mathscr{H}(b) = (bH^2)^\perp$.

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Non-extreme b

Definition

$b \in \mathbf{b}(H^\infty)$ is *non-extreme* if

$$\int_{\mathbb{T}} \log(1 - |b|)dm > -\infty.$$

Proposition

If $b \in \mathbf{b}(H^\infty)$ is non-extreme, then there is a unique outer a with $a(0) > 0$ and such that

$$|a|^2 + |b|^2 = 1$$

almost everywhere on $\mathbb{T}$.

Proposition

If $b \in \mathbf{b}(H^\infty)$ is non-extreme then $\mathscr{H}(b) \cap \mathcal{C}(\mathbb{D}^-)$ is dense in $\mathscr{H}(b)$.

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1. $\mathscr{M}(a) \subset \mathscr{H}(b)$ contractively.
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$$\inf_{z \in \mathbb{D}} (|a(z)| + |b(z)|) > 0 \quad \text{and} \quad T_{a/\overline{a}} \text{ is invertible}$$

Recall that $T_{a/\overline{a}}$ is invertible $\Leftrightarrow |a|^2$ is a Muckenhoupt weight.

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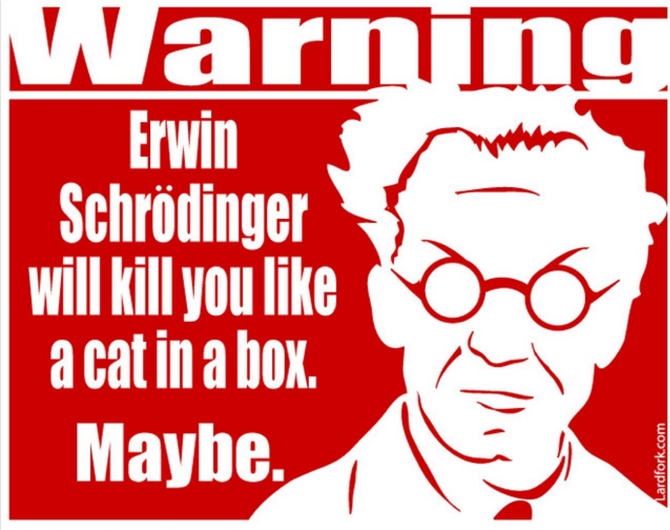
Warning
Erwin
Schrödinger
will kill you like
a cat in a box.
Maybe.
Lardfork.com

(Direct) Carleson embeddings for $\mathscr{H}(b)$

Direct embedding in $\mathscr{H}(b)$

Proposition (BFGHR (2013))

Suppose $b \in \mathbf{b}(H^\infty)$ is non-extreme and $\mu \in M_+(\mathbb{D}^-)$. If

$$\|f\|_\mu \lesssim \|f\|_b, \quad f \in \mathscr{H}(b) \cap \mathcal{C}(\mathbb{D}^-),$$

then $d\nu = |a|^2 d\mu$ is a Carleson measure for H^2.

Proof.

$$\|k_\lambda\|_2 = \|ak_\lambda\|_{\mathscr{M}(a)} \geq \|ak_\lambda\|_b \gtrsim \|ak_\lambda\|_\mu = \|k_\lambda\|_\nu.$$

□

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Direct embedding in $\mathscr{H}(b)$

Proposition (BFGHR (2013))

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For $b \in \mathbf{b}(H^\infty)$ and non-extreme and $\varepsilon > 0$, let

$$\Omega(b, \varepsilon) := \{z \in \mathbb{D} : |b(z)| < \varepsilon\}.$$

Let

$$\sigma(b) = \{\zeta \in \mathbb{T} : \varliminf_{z \to \zeta} |b(z)| = 0\}.$$

Theorem (Fricain-Baranov (2010))

If $\mu \in M_+(\mathbb{D}^-)$ such that

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(Reverse) Carleson embedding for $\mathscr{H}(b)$

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Reminder: $b \in \mathbf{b}(H^\infty)$ non-extreme $\Rightarrow \mathscr{H}(b) \cap \mathcal{C}(\mathbb{D}^-)$ dense in $\mathscr{H}(b)$.

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Proposition (BFGHR (2013))

Suppose $\|k_\lambda\|_b \lesssim \|k_\lambda\|_\mu$ for all $\lambda \in \mathbb{D}$. Then $b/a \in H^2$.

Proof.

$$\begin{aligned}
\|k_\lambda\|_b^2 &= \|k_\lambda\|_2^2 + \|T_{\overline{b/a}} k_\lambda\|_2^2 \\
&= \|k_\lambda\|_2^2 + |\frac{b}{a}(\lambda)|^2 \|k_\lambda\|_2^2 \\
&= \frac{1}{1-|\lambda|^2}(1 + |\frac{b}{a}(\lambda)|^2) \\
&\leq C \int \frac{1}{|1-\overline{\lambda} z|^2} d\mu(z)
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$$\left|\frac{b}{a}(\lambda)\right|^2 \leq C \int \frac{1-|\lambda|^2}{|1-\overline{\lambda} z|^2} d\mu(z)$$

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More about $b/a \in H^2$ and its consequences

Recall that $\|k_\lambda\|_b \lesssim \|k_\lambda\|_\mu, \lambda \in \mathbb{D} \Rightarrow b/a \in H^2$.

Proposition (Sarason)

If $b \in \mathbf{b}(H^\infty)$ and non-extreme then TFAE:

1. $b/a \in H^2$;
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This means that if $\|f\|_b \lesssim \|f\|_\mu$ for all $f \in \mathscr{H}(b)$, then

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Important consequence

WLOG assume $\mu|_{\mathbb{T}} \ll m$

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Reverse Carleson embedding for $\mathscr{H}(b)$

Theorem (BFGHR (2013))

Let $b \in \mathbf{b}(H^\infty)$ be non-extreme and $\mu \in M_+(\mathbb{D}^-)$ such that $\mu|_{\mathbb{T}} \ll m$. Let $h = d\mu|_{\mathbb{T}}/dm$. TFAE:

1. $\|f\|_b \lesssim \|f\|_\mu$ *holds for every* $f \in \mathscr{H}(b)$;
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Reverse Carleson embedding for $\mathscr{H}(b)$

Theorem (BFGHR (2013))

Let $b \in \mathbf{b}(H^\infty)$ be non-extreme and $\mu \in M_+(\mathbb{D}^-)$ such that $\mu|_{\mathbb{T}} \ll m$. Let $h = d\mu|_{\mathbb{T}}/dm$. TFAE:

1. $\|f\|_b \lesssim \|f\|_\mu$ *holds for every* $f \in \mathscr{H}(b)$;
2. $\|k_\lambda^b\|_b \lesssim \|k_\lambda^b\|_\mu$ *holds for every* $\lambda \in \mathbb{D}$.
3. $d\nu := |a|^2 d\mu$, *satisfies*

$$\inf_I \frac{\nu\left(S(I)\right)}{m(I)} > 0;$$

4. $\inf_{\zeta \in \mathbb{T}} |a(\zeta)|^2 h(\zeta) > 0.$

Embedding and Reverse embedding for $\mathscr{H}(b)$

Question

For $\mu \in M_+(\mathbb{D}^-)$ when do we have $\|f\|_b \asymp \|f\|_\mu$, $f \in \mathscr{H}(b)$?

Theorem

Let $b \in \mathbf{b}(H^\infty)$ be non-extreme and let $\mu \in M_+(\mathbb{D}^-)$ with $\mu|\mathbb{T} \ll m$ and let $d\nu = |a|^2 d\mu$. TFAE

1. $\|f\|_b \asymp \|f\|_\mu, \quad f \in \mathscr{H}(b)$;
2. *The following conditions hold:*
    1. $\inf\{|a(z)| + |b(z)| : z \in \mathbb{D}\} > 0$;
    2. $|a|^2$ *is a Muckenhaupt weight;*
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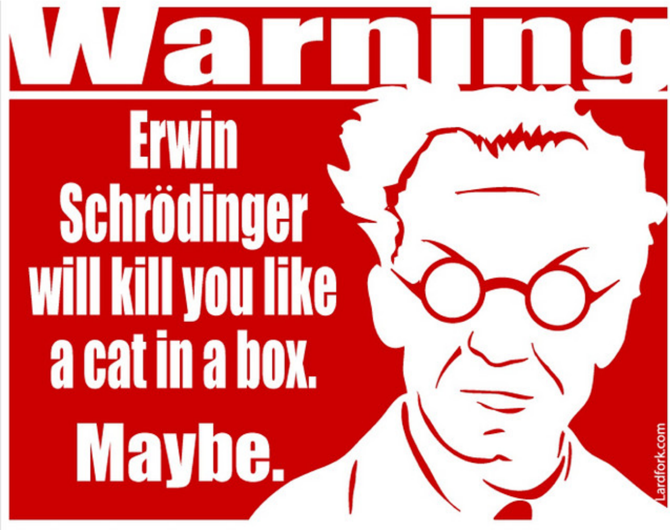
Example

Let

$$a(z) = c_\alpha(1 - z)^\alpha,$$

where $\alpha \in (0, 1/2)$ and b be the Pythagorean mate for a. Let

$$d\mu = \frac{1}{|a|^2} dm.$$


Warning
Erwin
Schrödinger
will kill you like
a cat in a box.
Maybe.
Lardfork.com

Isometric embeddings of $\mathscr{H}(b)$

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For $\mu \in M_+(\mathbb{D}^-)$, when do we have $\|f\|_b = \|f\|_\mu$ for all $f \in \mathscr{H}(b)$?

Theorem

For $b \in \mathbf{b}(H^\infty)$ non-extreme (and non-constant), there are no isometric measures for $\mathscr{H}(b)$.

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Isometric embeddings of $\mathscr{H}(b)$

Proposition

With $b \in \mathbf{b}(H^\infty)$ non-extreme (and non-constant) and $b/a \in H^2$ we have

$$\|z^n\|_b^2 = 1 + \sum_{j=0}^{n} |\widehat{b/a}(j)|^2.$$

Proof.

$$\|z^n\|_b^2 = \|z^n\|_2^2 + \|T_{\overline{b/a}} z^n\|_2^2$$

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Proof there are no isometric measures for $\mathscr{H}(b)$

Suppose $\|z^n\|_\mu = \|z^n\|_b$ for all n. Then

$$\|z^n\|_\mu^2 = \mu(\mathbb{T}) + \int_{\mathbb{D}} |z|^{2n} d\mu(z) = \|z^n\|_b^2 = 1 + \sum_{j=0}^{n} |\widehat{b/a}(j)|^2$$

Now let $n \to \infty$ to get (via DCT)

$$\mu(\mathbb{T}) = 1 + \sum_{j=0}^{\infty} |\widehat{b/a}(j)|^2.$$

$$\Rightarrow \sum_{j=n+1}^{\infty} |\widehat{b/a}(j)|^2 = 0$$

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$$1 = |a|^2 + |b|^2 = |a|^2(1 + |k|^2)$$

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What happens when b is constant?

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If b is constant, then $\mathscr{H}(b) = H^2$ with the same norm and the only isometric measure on H^2 is m.

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A brief remark on the Inner case

Dominating sets

A set $\Sigma \subset \mathbb{T}, m(\Sigma) < 1$, is *dominating* for $(\Theta H^2)^\perp$ if

$$\int_{\mathbb{T}} |f|^2 dm \lesssim \int_{\Sigma} |f|^2 dm, \quad f \in (\Theta H^2)^\perp.$$

Equivalently, if $d\mu = \chi_\Sigma dm$, then $\| \cdot \|_\mu \asymp \| \cdot \|_2$ on $(\Theta H^2)^\perp$.

Theorem (BFGHR (2012))

Every model space $(\Theta H^2)^\perp$ admits a dominating set Σ.

Proposition

If Σ is dominating for $(\Theta H^2)^\perp$ then

- $d(\Sigma, \{\xi : |\Theta'(\xi)| = +\infty\}) = 0$.
- $d(\Sigma, \sigma(\Theta)) = 0$.

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Reverse embeddings via dominating sets

Theorem (BFGHR (2012))

Let Θ be an inner function, Σ be a dominating set for $(\Theta H^2)^\perp$, and $\mu \in M_+(\mathbb{D}^-)$ be such that $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$. Suppose that

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where the infimum is taken over all arcs $I \subset \mathbb{T}$ for which $I \cap \Sigma \neq \varnothing$, then $\| \cdot \|_\mu \asymp \| \cdot \|_2$ on $(\Theta H^2)^\perp$.

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