Two weight theorems and a characterization for admissible local transforms

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Hilbert function spaces, Gargnano

May 23, 2013

Standard fractional singular kernels

- Let $0 \leq \alpha < n$. Consider a kernel function $K^\alpha(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the fractional size and smoothness conditions,

$$\begin{aligned} \left|K^\alpha(x,y)\right| &\leq C\left|x-y\right|^{\alpha-n}, && (1) \\ \left|K^\alpha(x,y) - K^\alpha\left(x',y\right)\right| &\leq C\frac{\left|x-x'\right|}{\left|x-y\right|}\left|x-y\right|^{\alpha-n}, && \frac{\left|x-x'\right|}{\left|x-y\right|} \leq \frac{1}{2}, \\ \left|K^\alpha(x,y) - K^\alpha\left(x,y'\right)\right| &\leq C\frac{\left|y-y'\right|}{\left|x-y\right|}\left|x-y\right|^{\alpha-n}, && \frac{\left|y-y'\right|}{\left|x-y\right|} \leq \frac{1}{2}. \end{aligned}$$

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- The Cauchy integral C^1 in the complex plane arises when $K(x, y) = \frac{1}{x-y}$, $x, y \in \mathbb{C}$. The fractional size and smoothness condition (1) holds with $n = 2$ and $\alpha = 1$ in this case.

Standard fractional singular integrals

Definition

We say that T^α is a *standard α-fractional integral operator with kernel K^α* if T^α is a bounded linear operator from some $L^p(\mathbb{R}^n)$ to some $L^q(\mathbb{R}^n)$ for some fixed $1 < p \leq q < \infty$, that is

$$\left\| T^\alpha f \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n)}, \qquad f \in L^p(\mathbb{R}^n),$$

if $K^\alpha(x,y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies (1), and if T^α and K^α are related by

$$T^\alpha f(x) = \int K^\alpha(x,y) f(y) dy, \qquad \text{a.e.-}x \notin \mathit{supp}\ f,$$

whenever $f \in L^p(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$. We say $K^\alpha(x,y)$ is a *standard α-fractional kernel* if it satisfies (1).

The basic problem

Given two locally finite positive Borel measures σ and ω on $\mathbb{R}^n$, and a standard α-fractional integral operator T, characterize the boundedness of T_σ from $L^2(\sigma)$ to $L^2(\omega)$:

$$\left(\int_{\mathbb{R}^n} |Tf\sigma|^2 \, d\omega\right)^{\frac{1}{2}} \leq \mathfrak{N}_T \left(\int_{\mathbb{R}^n} |f|^2 \, d\sigma\right)^{\frac{1}{2}}, \qquad f \in L^2(\sigma),$$

uniformly over all smooth truncations of the operator T.

- In 1984 David and Journé showed that if $K(x,y)$ is a standard kernel on $\mathbb{R}^n$,

$$\begin{aligned} |K(x,y)| &\leq C|x-y|^{-n}, \\ |K(x',y)-K(x,y)|+\dots &\leq C|x-y|^{-n}\left(\frac{|x'-x|}{|x-y|}\right)^{\delta}, \end{aligned}$$

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- and if $Tf(x) \equiv \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$ for $x \notin supp\ f$, then T is bounded on $L^2(\mathbb{R}^n)$ if and only if $T \in WBP$ and

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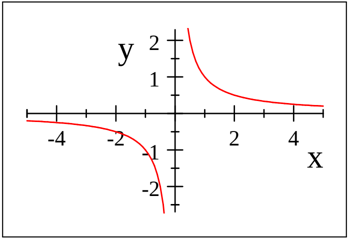
Definition ($T1$ or *testing* conditions)

$$\begin{aligned} T1 \in BMO \quad & \left(\Leftrightarrow \int_Q |T\chi_Q|^2 \leq C|Q|\right), \\ T^*1 \in BMO \quad & \left(\Leftrightarrow \int_Q |T^*\chi_Q|^2 \leq C|Q|\right). \end{aligned}$$

The Hilbert transform
as singular integral

The Hilbert transform Hf arose in 1905 in connection with Hilbert's twenty-first problem, and for $f \in L^2(\mathbb{R})$ is defined almost everywhere by the *principal value* singular integral

$$\begin{aligned} Hf(x) &= p.v. \int \frac{1}{y-x} f(y)\, dy \\ &\equiv \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \frac{1}{y-x} f(y)\, dy, \qquad a.e. x \in \mathbb{R}. \end{aligned}$$



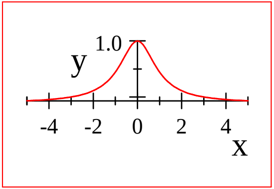
The convolution kernel of H

Toward a geometric characterization

- In 2004 Nazarov, Treil and Volberg showed that if a weight pair (ω, σ) satisfies the pivotal condition

$$\sum_{r=1}^{\infty} |I_r|_\omega \, \mathrm{P}(I_r, \chi_{I_0}\sigma)^2 \leq \mathcal{P}_*^2 \, |I_0|_\sigma \, ; \quad \mathrm{P}(I, \nu) = \int \frac{|I|}{|I|^2 + x^2} d\nu(x) \, ,$$

and its dual for all decompositions of an interval I_0 into subintervals I_r,



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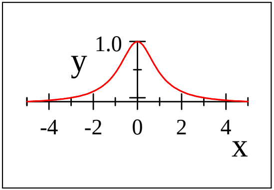

- then the Hilbert transform H satisfies the two weight L^2 inequality

$$\int |H(f\sigma)|^2 d\omega \leq C \int |f|^2 d\sigma,$$

uniformly for all smooth truncations of the Hilbert transform,

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Definition (A_2 condition on steroids)

$$\sup_I \mathrm{P}(I,\omega) \cdot \mathrm{P}(I,\sigma) \equiv \mathcal{A}_2^2 < \infty \,,$$

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Definition (A_2 condition on steroids)

$$\sup_I \mathrm{P}(I, \omega) \cdot \mathrm{P}(I, \sigma) \equiv \mathcal{A}_2^2 < \infty ,$$

- as well as the two *interval testing* conditions

$$\begin{aligned} \int_I \left| H(\chi_I \sigma) \right|^2 d\omega &\leq \mathfrak{T}^2 \left| I \right|_\sigma , \\ \int_I \left| H(\chi_I \omega) \right|^2 d\sigma &\leq \left(\mathfrak{T}^* \right)^2 \left| I \right|_\omega . \end{aligned}$$

The Nazarov Treil Volberg conjecture

A question raised in Volberg's 2003 CBMS book, known as the *NTV conjecture*, is whether or not

$$\int_{\mathbb{R}} \left|H\left(f\sigma\right)\right|^2 \omega \leq \mathfrak{N} \int_{\mathbb{R}} \left|f\right|^2 \sigma, \qquad f \in L^2\left(\sigma\right), \tag{2}$$

is equivalent to the $\mathcal{A}_2$ condition and the two interval testing conditions.

The indicator/interval NTV conjecture

Theorem (Lacey, Sawyer, Shen and Uriarte-Tuero (2012))

The best constant $\mathfrak{N}$ *in the two weight inequality (2) for the Hilbert transform satisfies*

$$\mathfrak{N} \approx \sqrt{\mathcal{A}_2} + \mathfrak{A} + \mathfrak{A}^*,$$

where $\mathfrak{A}, \mathfrak{A}^*$ *are the best constants in the indicator/interval testing conditions,*

$$\int_I \left| H\left(\mathbf{1}_E \sigma\right) \right|^2 \omega \leq \mathfrak{A} \left| I \right|_\sigma , \quad \int_I \left| H\left(\mathbf{1}_E \omega\right) \right|^2 \sigma \leq \mathfrak{A}^* \left| I \right|_\omega ,$$

for all intervals I *and closed subsets* E *of* I*. Note that* E *does not appear on the right side of these inequalities, and that if* H *were a positive operator we could take* $E = I$*.*

The NTV conjecture

In January 2013 M. Lacey found the final stopping time and recursion argument needed to finish the proof of the NTV conjecture.

Theorem (Lacey)

The best constant $\mathfrak{N}$ *in the two weight inequality (2) for the Hilbert transform satisfies*

$$\mathfrak{N} \approx \sqrt{\mathcal{A}_2} + \mathfrak{T} + \mathfrak{T}^*,$$

i.e. H_σ *is bounded from* $L^2(\sigma)$ *to* $L^2(\omega)$ *if and only if the strong* A_2 *and interval testing conditions hold.*

Difficulties in higher dimensions

Positive derivative of the kernel

- The arguments in dimension $n = 1$ are tied very closely to the *positivity* of the derivative $K'(x)$ of the Hilbert transform kernel $K(x) = -\frac{1}{x}$.

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- The arguments in dimension $n = 1$ are tied very closely to the *positivity* of the derivative $K'(x)$ of the Hilbert transform kernel $K(x) = -\frac{1}{x}$.
- Indeed, this property underlies the necessity of energy,

$$\sum_{r=1}^{\infty} |I_r|_{\omega} \mathsf{E}(I_r, \omega)^2 \mathrm{P}(I_r, \chi_{I_0}\sigma)^2 \leq \mathcal{E}^2 |I_0|_{\sigma}, \qquad I_0 = \dot{\bigcup}_{r=1}^{\infty} I_r ,$$

where

$$\mathsf{E}(J, \omega) \equiv \left(\mathbb{E}_J^{\omega(dx)} \mathbb{E}_J^{\omega(dx')} \left(\frac{|x - x'|}{|J|} \right)^2 \right)^{1/2},$$

Difficulties in higher dimensions

Necessity of energy

- The energy condition can be derived from the following elementary calculation for $-a \leq x' < x \leq a$:

$$\begin{aligned} H\nu\left(x\right) - H\nu\left(x'\right) &= \int_{\mathbb{R}\setminus[-a,a]} \left\{ \frac{1}{y-x} - \frac{1}{y-x'} \right\} d\nu\left(y\right) \\ &= \left(x - x'\right) \int_{\mathbb{R}\setminus[-a,a]} \frac{1}{\left(y-x\right)\left(y-x'\right)} d\nu\left(y\right) \\ &\geq \frac{1}{4}\left(x-x'\right) \int_{\mathbb{R}\setminus[-a,a]} \frac{1}{y^2} d\nu\left(y\right). \end{aligned}$$

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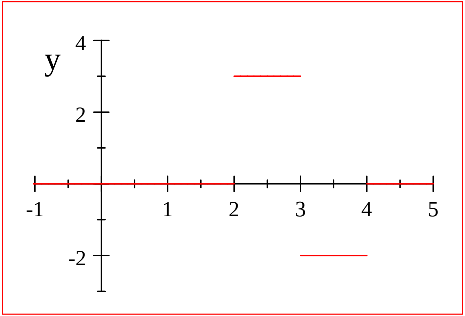
- The kernels of singular integrals in higher dimension no longer have such a positivity property, and this represents the major obstacle to extending the ideas of Nazarov-Treil-Volberg, Lacey-Sawyer-Shen-UriarteTuero and Lacey to dimension greater than one.

Haar functions adapted to a measure

- The Haar function h_I^σ adapted to a positive measure σ and a dyadic interval $I \in \mathcal{D}$ is a positive (negative) constant on the left (right) child, has vanishing mean $\int h_I^\sigma d\sigma = 0$, and is normalized $\|h_I^\sigma\|_{L^2(\sigma)} = 1$. For example if $|[2,3]|_\sigma = \frac{1}{15}$ and $|[3,4]|_\sigma = \frac{1}{10}$, then



The Haar function $h_{[2,4]}^\sigma$

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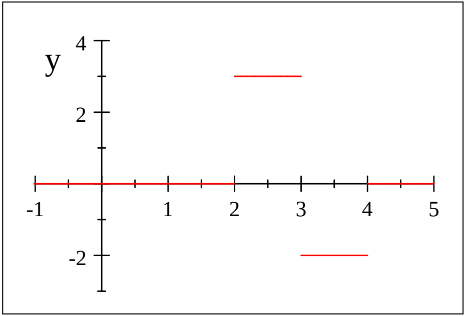

The Haar function $h^\sigma_{[2,4]}$

- The supremum norm of h_I^σ is quite large if σ is very unbalanced (not doubling).

Difficulties in higher dimensions

The monotonicity property

- The positivity of the derivative of the kernel $-\frac{1}{x}$ gives the Monotonicity Property involving the Haar function $h_I^\omega = \frac{1}{\gamma_\omega}\left(-\frac{1}{|I_-|_\omega}\mathbf{1}_{I_-} + \frac{1}{|I_+|_\omega}\mathbf{1}_{I_+}\right)$ and a signed measure ν satisfying $|\nu| \leq \mathbf{1}_{\mathbb{R}\setminus I}\mu$: namely $\langle H\nu, h_I^\omega\rangle_\omega$ equals

$$
\begin{aligned}
& \int_{I_+} H\nu(x)\, h_I^\omega(x)\, d\omega(x) + \int_{I_-} H\nu(x')\, h_I^\omega(x')\, d\omega(x') \\
= & \int_{I_+}\int_{I_-}\left[H\nu(x) - H\nu(x')\right]\left|h_I^\omega(x')\right| d\omega(x')\left|h_I^\omega(x)\right| d\omega(x) \\
= & \int_{I_+}\int_{I_-}\int_{\mathbb{R}\setminus I} \frac{x-x'}{(y-x)(y-x')} d\nu(y)\left|h_I^\omega(x')\right| d\omega(x')\left|h_I^\omega(x)\right| d\omega
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$$\begin{aligned} & \int_{I_+} H\nu(x)\, h_I^\omega(x)\, d\omega(x) + \int_{I_-} H\nu(x')\, h_I^\omega(x')\, d\omega(x') \\ = & \int_{I_+}\int_{I_-} \left[H\nu(x) - H\nu(x')\right] \left|h_I^\omega(x')\right| d\omega(x') \left|h_I^\omega(x)\right| d\omega(x) \\ = & \int_{I_+}\int_{I_-}\int_{\mathbb{R}\setminus I} \frac{x-x'}{(y-x)(y-x')} d\nu(y) \left|h_I^\omega(x')\right| d\omega(x') \left|h_I^\omega(x)\right| d\omega \end{aligned}$$

- and since $\frac{x-x'}{(y-x)(y-x')} \geq 0$ for $y \in \mathbb{R}\setminus I$ and $x \in I_+$ and $x' \in I_-$, we have $\left|\langle H\nu, h_I^\omega\rangle_\omega\right| \leq \langle H\mu, h_I^\omega\rangle_\omega$.

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$$\int_{I_+} H\nu(x)\, h_I^\omega(x)\, d\omega(x) + \int_{I_-} H\nu(x')\, h_I^\omega(x')\, d\omega(x')$$

$$= \int_{I_+}\int_{I_-}\left[H\nu(x) - H\nu(x')\right]\left|h_I^\omega(x')\right| d\omega(x')\left|h_I^\omega(x)\right| d\omega(x)$$

$$= \int_{I_+}\int_{I_-}\int_{\mathbb{R}\setminus I} \frac{x-x'}{(y-x)(y-x')} d\nu(y)\left|h_I^\omega(x')\right| d\omega(x')\left|h_I^\omega(x)\right| d\omega$$

- and since $\frac{x-x'}{(y-x)(y-x')} \geq 0$ for $y \in \mathbb{R}\setminus I$ and $x \in I_+$ and $x' \in I_-$, we have $\left|\langle H\nu, h_I^\omega\rangle_\omega\right| \leq \langle H\mu, h_I^\omega\rangle_\omega$.
- If $\operatorname{supp}\mu \subset \mathbb{R}\setminus 2I$, then $\langle H\mu, h_I^\omega\rangle_\omega \approx \frac{\mathrm{P}(I,\mu)}{|I|}\widehat{x}(I)$.

- If f is of minimal bounded fluctuation, then there is a collection $\mathcal{K}_f$ of pairwise disjoint subintervals of I such that

$$f=\sum_{I\in\pi\mathcal{K}_f}\widehat{f}(I)\ h_I^\sigma=\sum_{I\in\pi\mathcal{K}_f}\triangle_I^\sigma f,$$

where if $I=\pi K$, then $K=I_-$, the child of I with smallest σ-measure.

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- The key additional property, besides that of bounded fluctuation, of such an f is

$$\mathbb{E}_{I_+}^\sigma \triangle_I^\sigma f \geq 0, \qquad \text{for all } I \in \mathcal{K}_f.$$

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- This no longer holds in higher dimensions. However, the stopping time and recursion argument of Lacey circumvents the need for minimal bounded fluctuation, and is a very robust argument needing only the energy conditions, with no special properties of the Hilbert transform.

The main theorem

Theorem (Sawyer, Shen and Uriarte-Tuero)

Suppose that T^{α} is a standard α-fractional Calderón-Zygmund operator on $\mathbb{R}^n$, and that ω and σ are positive Borel measures on $\mathbb{R}^n$ without common point masses. Set $T^{\alpha}_{\sigma} f = T^{\alpha}(f\sigma)$ for any smooth truncation of T^{α}_{σ}. Suppose $0 \leq \alpha < n$. Then the operator T^{α}_{σ} is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.

$$\left\| T^{\alpha}_{\sigma} f \right\|_{L^2(\omega)} \leq \mathfrak{N}^{\alpha} \left\| f \right\|_{L^2(\sigma)}, \tag{3}$$

uniformly in smooth truncations of T^{α}, and moreover

$$\mathfrak{N}_{\alpha} \leq C_{\alpha} \left(\sqrt{\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,*}_2} + \mathfrak{T}_{\alpha} + \mathfrak{T}^{*}_{\alpha} + \mathcal{E}_{\alpha} + \mathcal{E}^{*}_{\alpha} \right),$$

provided that the following three conditions hold:

A2 and Testing conditions

- The two dual $\mathcal{A}_2^\alpha$ conditions hold,

$$\mathcal{A}_2^\alpha \equiv \sup_{Q\in\mathcal{Q}^n} \mathcal{P}^\alpha(Q,\sigma)\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} < \infty,$$

$$\mathcal{A}_2^{\alpha,*} \equiv \sup_{Q\in\mathcal{Q}^n} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^\alpha(Q,\omega) < \infty,$$

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\mathcal{A}_2^{\alpha,*} &\equiv \sup_{Q\in\mathcal{Q}^n} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^\alpha(Q,\omega) < \infty,
\end{aligned}$$

- and the two dual testing conditions hold,

$$\begin{aligned}
\mathfrak{T}_\alpha^2 &\equiv \sup_{Q\in\mathcal{Q}^n} \frac{1}{|Q|_\sigma}\int_Q \left|T^\alpha(\mathbf{1}_Q\sigma)\right|^2\omega < \infty, \\
(\mathfrak{T}_\alpha^*)^2 &\equiv \sup_{Q\in\mathcal{Q}^n} \frac{1}{|Q|_\omega}\int_Q \left|(T^\alpha)^*(\mathbf{1}_Q\omega)\right|^2\sigma < \infty,
\end{aligned}$$

Energy conditions

and the two dual energy conditions hold,

$$\left(\mathcal{E}_{\alpha}\right)^{2} \equiv \sup_{\substack{Q=\dot{\cup}Q_r \\ Q,Q_r\in\mathcal{Q}^n}} \frac{1}{|Q|_{\sigma}} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^{\alpha}\left(Q_r, \mathbf{1}_{Q\setminus Q_r}\sigma\right)}{|Q_r|} \right)^{2} \left\| \widetilde{\mathrm{P}}_{Q_r}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^{2} < \infty,$$

$$\left(\mathcal{E}_{\alpha}^{*}\right)^{2} \equiv \sup_{\substack{Q=\dot{\cup}Q_r \\ Q,Q_r\in\mathcal{Q}^n}} \frac{1}{|Q|_{\omega}} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^{\alpha}\left(Q_r, \mathbf{1}_{Q\setminus Q_r}\omega\right)}{|Q_r|} \right)^{2} \left\| \widetilde{\mathrm{P}}_{Q_r}^{\sigma} \mathbf{x} \right\|_{L^2(\sigma)}^{2} < \infty,$$

where the goodness parameters r and ε implicit in the definition of $\widetilde{\mathrm{P}}$ are fixed sufficiently large and small respectively depending on dimension, and the two inequalities hold uniformly over all dyadic grids. The differing Poisson kernels are defined below.

Necessity of A2

- Conversely, suppose $0 \leq \alpha < n$ and that $\left\{T_j^\alpha\right\}_{j=1}^J$ is a collection of Calderón-Zygmund operators with standard kernels $\left\{K_j^\alpha\right\}_{j=1}^J$. In the range $0 \leq \alpha < \frac{n}{2}$, we assume there is $c > 0$ such that for *each* unit vector $\mathbf{u}$ there is j satisfying

$$\left|K_j^\alpha\left(x, x + t\mathbf{u}\right)\right| \geq ct^{\alpha - n}, \qquad t \in \mathbb{R}. \tag{4}$$

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- For the range $\frac{n}{2} \leq \alpha < n$, we asume that for each $m \in \{1, -1\}^n$, there is a sequence of coefficients $\left\{ \lambda_j^m \right\}_{j=1}^J$ such that

$$\left| \sum_{j=1}^J \lambda_j^m K_j^\alpha \left(x, x + t\mathbf{u} \right) \right| \geq c t^{\alpha - n}, \qquad t \in \mathbb{R}. \tag{5}$$

holds for *all* unit vectors $\mathbf{u}$ in the n-ant

$$V_m = \left\{ x \in \mathbb{R}^n : m_i x_i > 0 \text{ for } 1 \leq i \leq n \right\}, \qquad m \in \{1, -1\}^n .$$

Necessity of A2 continued

- Furthermore, assume that each operator T_j^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$,

$$\left\| T_\sigma^\alpha f \right\|_{L^2(\omega)} \leq \mathfrak{N}_\alpha \left\| f \right\|_{L^2(\sigma)} .$$

Necessity of A2 continued

- Furthermore, assume that each operator T_j^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$,
$$\left\| T_\sigma^\alpha f \right\|_{L^2(\omega)} \leq \mathfrak{N}_\alpha \left\| f \right\|_{L^2(\sigma)} .$$
- Then the fractional $\mathcal{A}_2^\alpha$ condition holds, and moreover,
$$\sqrt{\mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,*}} \leq C\mathfrak{N}_\alpha .$$

Necessity of energy

- Conversely, suppose $\mathbf{n} = \mathbf{2}$ and $0 \leq \alpha < 2$, and that the $\mathcal{A}_2^\alpha$ condition holds, $\mathcal{A}_2^\alpha < \infty$, and that the dual cube testing conditions for an α-fractional admissible local transform vector $\mathbf{T}_M^\alpha$ hold,

$$\begin{aligned} \mathfrak{T}_{\mathbf{T}_M^\alpha}^2 &= \sup_{Q \in \mathcal{Q}^n} \frac{1}{|Q|_\sigma} \int_Q \left| \mathbf{T}_M^\alpha \left(\mathbf{1}_Q \sigma \right) \right|^2 \omega < \infty, \\ \left(\mathfrak{T}_{\mathbf{T}_M^\alpha}^* \right)^2 &= \sup_{Q \in \mathcal{Q}^n} \frac{1}{|Q|_\omega} \int_Q \left| \mathbf{T}_M^\alpha \left(\mathbf{1}_Q \omega \right) \right|^2 \sigma < \infty. \end{aligned}$$

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- Then, provided the goodness parameters r and ε are fixed sufficiently large and small respectively depending on dimension, the two dual energy conditions hold, and moreover,

$$\mathcal{E}_\alpha + \mathcal{E}_\alpha^* \leq C \left(\sqrt{\mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,*}} + \mathfrak{T}_{\mathbf{T}_M^\alpha} + \mathfrak{T}_{\mathbf{T}_M^\alpha}^* \right).$$

NTV theorem for local transforms

We have the following generalization of the NTV conjecture to fractional admissible local vector transforms in dimension $n = 2$ with $0 \leq \alpha < 1$.

Corollary

Suppose $n = 2$ and $0 \leq \alpha < 1$. An α-fractional admissible local vector transform $\mathbf{T}_M^\alpha = \left(T_1^\alpha, \ldots, T_{2^M}^\alpha \right)$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the fractional $\mathcal{A}_2^\alpha$ condition holds, i.e. $\mathcal{A}_2^\alpha < \infty$, and the dual cube testing conditions for the fractional admissible local transform vector $\mathbf{T}_M^\alpha$ hold, i.e. $\mathfrak{T}_{\mathbf{T}_M^\alpha} + \mathfrak{T}_{\mathbf{T}_M^\alpha}^* < \infty$.

Admissible local transforms

- Define the rough *teepee* function $\Lambda_{K,rough}$ associated with an interval or arc K in the circle $\mathbb{S}^1$ as the unique function satisfying the three properties that $\Lambda_{K,rough}$

- vanish outside K,
- take the value 1 at the center of K,
- and be affine on both of the children $K_\pm$ of K.

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- Then a *smooth* teepee function Λ_K on K is a smooth function on the circle supported in K and such that

$$\sup_{\theta \in \mathbb{S}^1} \left| \Lambda_K(\theta) - \Lambda_{K,rough}(\theta) \right| < \varepsilon(|K|),$$

where $\epsilon(|K|)$ is a sufficiently small number depending on the length $|K|$ of the interval K.

Admissible local transforms continued

- Given a large positive integer m, let $I = \left[-\frac{2\pi}{2^m}, \frac{2\pi}{2^m}\right)$ and set Ω to be $\Lambda_I - \Lambda_{I+\pi}$, where Λ_I is a smooth teepee function on I and $\Lambda_{I+\pi}(\theta) = \Lambda_I(\theta + \pi)$ is the rotation of Λ_I by angle π.

Admissible local transforms continued

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- Then with $M > m$ sufficiently large, we say that the collection of rotated functions

$$\{\Omega_\ell\}_{\ell=1}^{2^M}; \qquad \Omega_\ell(\theta) = \Omega\left(\theta - \frac{2\pi\ell}{2^M}\right),$$

and the corresponding vector of odd convolution α-fractional singular integrals,

$$\mathbf{T}_M^\alpha \equiv \{T_\ell^\alpha\}_{\ell=1}^{2^M}; \qquad T_\ell^\alpha(x) = \frac{\Omega_\ell(x)}{|x|^{2-\alpha}},$$

is admissible provided $m < M$ are taken large enough and $\epsilon(|K|) > 0$ is taken small enough.

Poisson integrals

- In higher dimensions, there are two natural 'Poisson integrals' P and $\mathcal{P}$ that arise, the usual Poisson integral P that emerges in connection with energy considerations, and a much smaller 'reproducing' Poisson integral $\mathcal{P}$ that emerges in connection with size considerations - in dimension $n = 1$ these two Poisson integrals coincide. For $0 \leq \alpha < n$, any cube Q and any positive Borel measure μ, let

$$\mathrm{P}^{\alpha}(Q, \mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}} + |x - x_Q|\right)^{n+1-\alpha}} d\mu(x),$$

$$\mathcal{P}^{\alpha}(Q, \mu) \equiv \int_{\mathbb{R}^n} \left(\frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}} + |x - x_Q|\right)^{2}}\right)^{n-\alpha} d\mu(x).$$

Poisson integrals continued

- Note that
- for $0 \leq \alpha < n-1$, P^{α} is strictly larger that $\mathcal{P}^{\alpha}$,
- for $\alpha = n-1$, P^{α} and $\mathcal{P}^{\alpha}$ coincide,
- for $n-1 < \alpha < n$, P^{α} is strictly smaller that $\mathcal{P}^{\alpha}$.

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- for $0 \leq \alpha < n-1$, P^{α} is strictly larger that $\mathcal{P}^{\alpha}$,
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- for $n-1 < \alpha < n$, P^{α} is strictly smaller that $\mathcal{P}^{\alpha}$.
- The standard Poisson integral P^{α} appears in the energy conditions, while the reproducing Poisson kernel $\mathcal{P}^{\alpha}$ appears in the $\mathcal{A}_2^{\alpha}$ conditions.

The good dyadic grids of NTV in dimension one

- For any $\beta = \{\beta_l\} \in \{0,1\}^{\mathbb{Z}}$, define the dyadic grid $\mathbb{D}_\beta$ to be the collection of intervals
$$\mathbb{D}_\beta = \left\{ 2^n \left([0,1) + k + \sum_{i<n} 2^{i-n} \beta_i \right) \right\}_{n\in\mathbb{Z},\, k\in\mathbb{Z}}$$
and place the usual uniform probability measure $\mathbb{P}$ on the space $\{0,1\}^{\mathbb{Z}}$.

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- For weights ω and σ, consider random choices of dyadic grids $\mathcal{D}^\omega$ and $\mathcal{D}^\sigma$. Fix $\varepsilon > 0$ and for a positive integer r, an interval $J \in \mathcal{D}^\omega$ is said to be *r-bad* if there is an interval $I \in \mathcal{D}^\sigma$ with $|I| \geq 2^r |J|$, and

$$\mathrm{dist}(e(I), J) \leq \tfrac{1}{2} |J|^\varepsilon |I|^{1-\varepsilon}.$$

where $e(I)$ is the set of the three discontinuities of h_I^σ. Otherwise, J is said to be *r-good*.

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  where $e(I)$ is the set of the three discontinuities of h_I^σ. Otherwise, J is said to be *r-good*.
- We have

$$\mathbb{P}(J \text{ is } r\text{-bad}) \leq C 2^{-\varepsilon r}.$$

Energy conditions

- Define P_I^{μ} to be orthogonal projection onto the subspace of $L^2(\mu)$ consisting of functions supported in I with μ-mean value zero.

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- In addition, define $\widetilde{\mathsf{P}}_I^\mu$ to be orthogonal projection onto the subspace $L^2_{\mathcal{H}(I)}(\mu)$ of $L^2(\mu)$ consisting of those functions $f \in L^2(\mu)$ whose Haar support is contained in

$$\mathcal{H}(I) \equiv \left\{ J \in \mathcal{D}^n : \textbf{either } J \subset I \text{ and } |J|^{\frac{1}{n}} > 2^{-r} |I|^{\frac{1}{n}} \textbf{ or } J \Subset I \right\},$$

and where the notation $J \Subset I$, read J is deeply embedded in I, means that $J \subset I$, $|J|^{\frac{1}{n}} \leq 2^{-r} |I|^{\frac{1}{n}}$, and that J satisfies the 'good' condition relative to the cube I:

$$\operatorname{dist}(J, \partial I) > \frac{1}{2} |J|^{\frac{\varepsilon}{n}} |I|^{\frac{1-\varepsilon}{n}} .$$

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$$\operatorname{dist}(J, \partial I) > \frac{1}{2} |J|^{\frac{\varepsilon}{n}} |I|^{\frac{1-\varepsilon}{n}}.$$

- Here $r \in \mathbb{N}$ and $0 < \varepsilon < 1$ are the parameters in the definition of the 'good' dyadic grid below, and will be taken sufficiently large and small respectively depending on the dimension n.

Energy conditions continued

- In dimension $n = 1$ for $\alpha = 0$, we defined the energy condition by

$$\sum_{I \supset \dot{\cup} I_r} |I_r|_\omega \mathsf{E}(I_r, \mu)^2 \mathrm{P}^\alpha (I_r, \mathbf{1}_I \sigma)^2 \leq (\mathcal{E}_2)^2 |I|_\sigma ,$$

$$\mathsf{E}(I, \mu)^2 \equiv \frac{1}{|I|_\omega} \left\| \mathsf{P}_I^\mu \frac{\mathbf{x}}{|I|} \right\|_{L^2(\mu)}^2 .$$

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$$\mathrm{E}(I, \mu)^2 \equiv \frac{1}{|I|_\omega} \left\| \mathsf{P}_I^\mu \frac{\mathbf{x}}{|I|} \right\|_{L^2(\mu)}^2 .$$

- The extension of the *energy conditions* to higher dimensions will use the smaller projection $\widetilde{\mathsf{P}}_I^\mu \mathbf{x}$ in place of $\mathsf{P}_I^\mu \mathbf{x}$, and as a result, it is convenient to define the *soft* energy of μ on a cube J by

$$\mathrm{E}_{soft}(I, \mu)^2 \equiv \frac{1}{|I|_\omega} \left\| \widetilde{\mathsf{P}}_I^\mu \frac{\mathbf{x}}{|I|^{\frac{1}{n}}} \right\|_{L^2(\mu)}^2 .$$

Energy conditions continued

- Thus $\mathsf{E}_{soft}\left(I,\mu\right)$ includes precisely those Haar coefficients $\left\langle \mathbf{x}, h_J^{\omega,a}\right\rangle_\omega$ for which J is either close to I or deeply embedded in I. In particular, $\mathsf{E}_{soft}\left(I,\mu\right)$ includes all of the Haar coefficients $\left\langle \mathbf{x}, h_J^{\omega,a}\right\rangle_\omega$ for which J is good and contained in I, plus others.

Energy conditions continued

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- Then we define the forward energy condition in dimension $n \geq 2$ for $0 \leq \alpha < n$ by

$$\sum_{I \supset \dot{\cup} I_r} |I_r|_\omega \, \mathrm{E}_{soft}(I_r,\mu)^2 \, \mathrm{P}^\alpha(I_r, \mathbf{1}_I \sigma)^2 \leq \left(\mathcal{E}_2^\alpha\right)^2 |I|_\sigma .$$

  Note that this definition of the energy condition depends on the choice of goodness parameters r and ε.

Definition

Let $\mathcal{F}$ be a collection of dyadic cubes satisfying a Carleson condition

$$\sum_{F\in\mathcal{F}:\ F\subset S} |F|_{\sigma} \leq C_{\mathcal{F}} |S|_{\sigma}, \qquad S\in\mathcal{F},$$

where $C_{\mathcal{F}}$ is referred to as the Carleson norm of $\mathcal{F}$. A collection of functions $\{g_F\}_{F\in\mathcal{F}}$ in $L^2(w)$ is said to be $\mathcal{F}$*-adapted* if there are collections of cubes $\mathcal{J}(F) \subset \{J\in\mathcal{D}^{\sigma} : J \Subset F\}$, with $\mathcal{J}^*(F)$ consisting of the *maximal* dyadic cubes in $\mathcal{J}(F)$, such that the following three conditions hold:

F-adapted conditions

Definition

1. for each $F \in \mathcal{F}$, the Haar coefficients $\widehat{g_F}(J) = \langle g_F, h_J^\omega \rangle_\omega$ of g_F are nonnegative and supported in $\mathcal{J}(F)$, i.e.

$$\begin{cases} \widehat{g_F}(J) \geq 0 & \text{for all} \quad J \in \mathcal{J}(F) \\ \widehat{g_F}(J) = 0 & \text{for all} \quad J \notin \mathcal{J}(F) \end{cases}, \qquad F \in \mathcal{F},$$

2. the collection of sets of cubes $\{\mathcal{J}(F)\}_{F \in \mathcal{F}}$ is pairwise disjoint,
3. and there is a positive constant C such that if $\mathcal{J}^*(F)$ consists of the maximal cubes in $\mathcal{J}(F)$, then for every cube I in $\mathcal{D}^\sigma$, the set of pairs of cubes (F, J^*) that 'straddle' I,

$$\mathcal{B}_I \equiv \{(F, J^*) : J^* \in \mathcal{J}^*(F) \text{ and } J^* \subset I \subset F\},$$

satisfies the overlap condition $\sum_{(F,J^*) \in \mathcal{B}_I} \mathbf{1}_{J^*} \leq C, \qquad I \in \mathcal{D}^\sigma.$

The functional energy condition

- The *functional energy condition* is:

Definition

Let $\mathfrak{F}$ be the smallest constant in the inequality below, holding for all non-negative $h \in L^2(\sigma)$, all σ-Carleson collections $\mathcal{F}$, and all $\mathcal{F}$-adapted collections $\{g_F\}_{F\in\mathcal{F}}$:

$$\sum_{F\in\mathcal{F}} \sum_{J^*\in\mathcal{J}^*(F)} \mathrm{P}(J^*, h\sigma) \left| \left\langle \frac{x}{|J^*|}, g_F \mathbf{1}_{J^*} \right\rangle_\omega \right| \leq \mathfrak{F} \|h\|_{L^2(\sigma)} \left[\sum_{F\in\mathcal{F}} \|g_F\|^2_{L^2(\omega)} \right]^{1/2} . \tag{6}$$

Here $\mathcal{J}^*(F)$ consists of the *maximal* intervals J in the collection $\mathcal{J}(F)$.

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Here $\mathcal{J}^*(F)$ consists of the *maximal* intervals J in the collection $\mathcal{J}(F)$.

- The dual version of this condition has constant $\mathfrak{F}^*$.

Equivalence of functional energy and energy

Now we show that the functional energy constants are equivalent to the energy constants modulo $\mathcal{A}_2^\alpha$. First we use the two weight Poisson characterization to obtain

Lemma

$$\mathfrak{F}_\alpha \lesssim \mathcal{E}_\alpha + \sqrt{\mathcal{A}_2^\alpha} \text{ and } \mathfrak{F}_\alpha^* \lesssim \mathcal{E}_\alpha^* + \sqrt{\mathcal{A}_2^{\alpha,*}} \ .$$

Then we use an easy duality argument to show that

Lemma

$$\mathcal{E}_\alpha \lesssim \mathfrak{F}_\alpha \text{ and } \mathcal{E}_\alpha^* \lesssim \mathfrak{F}_\alpha^* \ .$$

- To prove the first lemma we fix $\mathcal{F}$ and set

$$\mu \equiv \sum_{F\in\mathcal{F}} \sum_{J^*\in\mathcal{J}^*(F)} \left\| \mathsf{P}^{\omega}_{F,J^*} \frac{x}{|J^*|} \right\|^2_{L^2(\omega)} \cdot \delta_{(c(J^*),|J^*|)} \ , \tag{7}$$

where the projections $\mathsf{P}^{\omega}_{F,J^*}$ onto Haar functions are defined by

$$\mathsf{P}^{\omega}_{F,J^*} \equiv \sum_{J\subset J^*:\ \pi_{\mathcal{F}}J=F} \triangle^{\omega}_{J} .$$

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- Here δ_q denotes a Dirac unit mass at a point q in the upper half plane $\mathbb{R}^2_+$. Note that we can replace x by $x-c$ for any choice of c we wish.

Two weight Poisson inequality

- We prove the two-weight inequality

$$\|\mathbb{P}(f\sigma)\|_{L^2(\mathbb{R}^2_+,\mu)} \lesssim \|f\|_{L^2(\sigma)}, \tag{8}$$

for all nonnegative f in $L^2(\sigma)$, noting that $\mathcal{F}$ and f are *not* related here.

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for all nonnegative f in $L^2(\sigma)$, noting that $\mathcal{F}$ and f are *not* related here.

- Above, $\mathbb{P}(\cdot)$ denotes the Poisson extension to the upper half-plane, so that in particular

$$\left\|\mathbb{P}(f\sigma)\right\|^2_{L^2(\mathbb{R}^2_+,\mu)} = \sum_{F\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(F)} \mathbb{P}(f\sigma)\left(c(J^*),|J^*|\right)^2 \left\|\mathsf{P}^{\omega}_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)}$$

and so (8) implies (6) by the Cauchy-Schwarz inequality.

Reduction to Poisson tent testing

By the two-weight inequality for the Poisson operator, inequality (8) requires checking these two inequalities

$$\int_{\mathbb{R}^2_+} \mathbb{P}\left(\mathbf{1}_I \sigma\right)(x,t)^2 \, d\mu(x,t) \equiv \left\|\mathbb{P}\left(\mathbf{1}_I \sigma\right)\right\|^2_{L^2(\widehat{I},\mu)} \lesssim \left(A_2 + \mathfrak{T}^2\right)\sigma(I), \quad (9)$$

$$\int_{\mathbb{R}} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2 \sigma(dx) \lesssim \mathcal{A}_2 \int_{\widehat{I}} t^2 \mu(dx,dt), \quad (10)$$

for all *dyadic* intervals $I \in \mathcal{D}$, where $\widehat{I} = I \times [0,|I|]$ is the box over I in the upper half-plane, and

$$\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu) = \int_{\widehat{I}} \frac{t^2}{t^2 + |x-y|^2} \mu(dy,dt).$$

Checking the Poisson testing conditions

- The main technical lemma used in proving (9) is this.

Lemma

We have

$$\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{M}(F)}\left(\frac{\mathrm{P}^\alpha\left(J^*,\mathbf{1}_{I\setminus F}\sigma\right)}{\left|J^*\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}^\omega_{F,J^*}x\right\|^2_{L^2(\omega)}\lesssim\mathcal{E}^2_\alpha\sigma(I)\,. \tag{11}$$

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- The proof is by duality and uses that the collection $\mathcal{F}$ satisfies a Carleson condition, hence has geometric decay in generations:

$$\sum_{F\in\mathcal{F}_I:\ d(F)=k}|F|_\sigma\lesssim 2^{-\delta k}\,|I|_\sigma\ ,\qquad k\geq 0,$$

which permits summing up energy condition estimates over generations.

Lemma (Local Reverse Energy)

Suppose that I and J are squares in $\mathbb{R}^2$ such that $\gamma J \subset I$, and that μ is a positive measure on $\mathbb{R}^2$ supported outside I. Suppose that $\left\{T_\ell^\alpha\right\}_{\ell=1}^{2^m}$ is as above with $0 < \alpha < 2$. Then for $\gamma > 1$ sufficiently large, and $\eta = \frac{2\pi}{2^m}$ sufficiently small, and $M > m$ sufficiently large, and $\epsilon\left(|K|\right) = \epsilon\left(2\eta\right)$ sufficiently small, we have the estimate

$$\mathsf{E}\left(J,\omega\right)^2 \mathrm{P}^\alpha\left(J,\mu\right)^2 \lesssim C_{\eta,\gamma}\mathbb{E}_J^{d\omega(x)}\mathbb{E}_J^{d\omega(z)}\left|\mathbf{T}_M^\alpha\mu\left(x\right)-\mathbf{T}_M^\alpha\mu\left(z\right)\right|^2,$$

where

$$\begin{aligned}\mathrm{P}^\alpha\left(J,\mu\right) &\approx \int \frac{\left|J\right|^{\frac{1}{n}}}{\left|y-c_J\right|^{n+1-\alpha}}d\mu\left(y\right),\\ c_J &= \left(c_J^1,\dots,c_J^n\right) \text{ is the center of } J.\end{aligned}$$

Necessity of weak energy

For $0 \leq \alpha < 1$ we have $\mathcal{E}_{\alpha}^{weak} \lesssim \mathfrak{T}_{\mathbf{T}_M^\alpha}$. Indeed, using local reverse energy with $\mu = \mathbf{1}_{I \setminus J_r^{**}} \sigma$, we 'plug the hole' in $I \setminus J_r^{**}$ to obtain

$$
\begin{aligned}
& \sum_{r=1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \left(\frac{\mathrm{P}^{\alpha}\left(J_r^{**}, \mathbf{1}_{I \setminus J_r^{**}} \sigma\right)}{\left|J_r^{**}\right|^{\frac{1}{n}}} \right)^2 \left\| \mathrm{P}_{J_r^*}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2 \\
\lesssim\ & \sum_{r=1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \int_{J_r^*} \left| \mathbf{T}_M^{\alpha} \mathbf{1}_{I \setminus J_r^{**}} \sigma \right|^2 d\omega \\
\lesssim\ & \sum_{r=1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \int_{J_r^*} \left| \mathbf{T}_M^{\alpha} \mathbf{1}_{I} \sigma \right|^2 d\omega + \sum_{r=1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \int_{J_r^*} \left| \mathbf{T}_M^{\alpha} \mathbf{1}_{J_r^{**}} \sigma \right|^2 d\omega \\
\lesssim\ & \int_I \left| \mathbf{T}_M^{\alpha} \mathbf{1}_{I} \sigma \right|^2 d\omega + \sum_{r=1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \int_{J_r^{**}} \left| \mathbf{T}_M^{\alpha} \mathbf{1}_{J_r^{**}} \sigma \right|^2 d\omega \\
\lesssim\ & \mathfrak{T}_{\mathbf{T}_M^\alpha} \left| I \right|_{\sigma} + \sum_{r=1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \mathfrak{T}_{\mathbf{T}_M^\alpha} \left| J_r^{**} \right|_{\sigma} \lesssim \mathfrak{T}_{\mathbf{T}_M^\alpha} \left| I \right|_{\sigma} \ ,
\end{aligned}
$$

Necessity of the energy condition

For $0 \leq \alpha < 1$ we have the energy condition $\mathcal{E}_\alpha \lesssim \mathfrak{T}_{\mathbf{T}_M^\alpha} + \sqrt{A_2^\alpha}$. Indeed,

$$
\begin{aligned}
& \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^\alpha \left(I_r, \mathbf{1}_I \sigma \right)}{|I_r|^{\frac{1}{n}}} \right)^2 \sum_{J \in \mathcal{H}(I_r)} \widehat{X}^\omega (J)^2 \\
\lesssim \; & \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^\alpha \left(I_r, \mathbf{1}_{I \setminus I_r} \sigma \right)}{|I_r|^{\frac{1}{n}}} \right)^2 \sum_{J \in \mathcal{H}(I_r)} \widehat{X}^\omega (J)^2 + OK \\
\lesssim \; & \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{J^* \in \mathcal{M}(I_r)} \left(\frac{\mathrm{P}^\alpha \left(J^*, \mathbf{1}_{I \setminus I_r} \sigma \right)}{|J^*|^{\frac{1}{n}}} \right)^2 \left(\sum_{J \subset J^*} \widehat{X}^\omega (J)^2 \right) + OK \\
\lesssim \; & \left(\mathcal{E}_\alpha^{weak} \right)^2 + A_2^\alpha \lesssim \left(\mathfrak{T}_{\mathbf{T}_M^\alpha} \right)^2 + A_2^\alpha .
\end{aligned}
$$

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