WEAK PRODUCT SPACES

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joint work
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Nehari's Theorem and H^1 - BMOA Duality

$[H^p = H^p(\mathbb{D})]$

For $\bar{b} \in H^2$ define the Hankel operator $H_b : H^2 \to \overline{H^2}$ by $H_b f = P_{\leq 0}(bf)$?

If $g \in H^2$ we get

$$\langle H_b f \mid \bar{g} \rangle_{H^2} = \langle bf \mid \bar{g} \rangle_{L^2}$$

$$= \int bfg \, \frac{|dz|}{2\pi} \, .$$

$= T_b(f, g)$

where $T_b : H^2 \times H^2 \to \mathbb{C}$?

is the Hankel form associated with b.

We see then that both ?'s have the same answer, i.e. that T_b is bounded iff H_b is, with equality of norms.

T_b bounded:

$$\left|\int b f g \frac{|dz|}{2\pi}\right| \leq C \|f\|_{H^2} \|g\|_{H^2}$$

But $H^1 = H^2 \cdot H^2$ with equality of norms, i.e.

$h \in H^1$ iff $h = fg$ where

$f, g \in H^2$ and $\|h\|_{H^1} = \|f\|_{H^2} \|g\|_{H^2}$

So above inequality becomes

$$\left|\int b h \frac{|dz|}{2\pi}\right| \leq C \|h\|_{H^1}$$

$\forall h \in H^1$

Hahn-Banach + $(L^1)^* = L^\infty$

$\Rightarrow \exists\, \varphi \in L^\infty$, $\|\varphi\|_{L^\infty} \leq C$.

such that

$$\int \varphi h \frac{|dz|}{2\pi} = \int b h \frac{|dz|}{2\pi} \quad \forall\, h \in H^1$$

But this means

$$b = P_{\le 0} \varphi$$

Fefferman-Stein informs us that

this is same as saying $b \in \overline{BMOA}$.

So $(H^1)^* = \overline{BMOA}$ with

$\langle \cdot | \cdot \rangle_{H^2}$ - pairing

~~How~~ How about in $\mathbb{B}_d$ for $d>1$?

Same definition of H_b, T_b, same argument as before, except now $H^1 \neq H^2 \cdot H^2$

BUT

Coifman, Rochberg, Weiss 1976 show <u>weak factorization</u>:

$h \in H^1$

Then $h = \sum f_i g_i$ where $f_i, g_i \in H^2$

and $\sum \|f_i\|_{H^2} \|g_i\|_{H^2} \leq C \|h\|_{H^1}$,

and this allows argument to go through. We similarly get

$(H^1)^* = BMO$ (where BMO defined using the usual "balls" in $\partial\mathbb{B}_d$)

How about polydisk $\mathbb{D}^d$ for $d > 1$?

$d=2$ Ferguson, Lacey 2002

$d>2$ Lacey, Terwilliger 2009

show weak factorization for H^1, and it follows that

$$(H^1)^* = BMOA$$

where now BMOA is the Chang-Fefferman version.

General Scheme

$\mathcal{H}$ a Hilbert space of analytic functions on a domain $\Omega \subseteq \mathbb{C}^d$, i.e. $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ continuously.

$\mathcal{K}$ a Banach space of analytic functions on Ω such that

$$f, g \in \mathcal{H} \Rightarrow fg \in \mathcal{K}$$

[necessarily then $\|fg\|_{\mathcal{K}} \leq \tilde{C} \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}$

and if $h \in \mathcal{K}$ $\exists$ $f_i, g_i \in \mathcal{H}$ such that

$h = \sum f_i g_i$ with $f_i, g_i \in \mathcal{D}$ and

$$\sum \|f_i\|_{\mathcal{D}} \|g_i\|_{\mathcal{D}} \leq C \|h\|_{\mathcal{K}}$$

— let's also throw in $1 \in \mathcal{H}$

With $T_b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ for $b \in \mathcal{H}$

defined as before, $T_b(f,g) = \langle fg \mid b \rangle_{\mathcal{H}}$,

define

$$\mathcal{X}(\mathcal{H}) = \{ b \in \mathcal{H} : |\langle T_b(f,g) \rangle| \le C \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}} \}$$

with $\|b\|_{\mathcal{X}(\mathcal{H})} = \inf C$'s

Then $\mathcal{X}(\mathcal{H}) = \mathcal{K}^*$ with $\langle \cdot | \cdot \rangle_{\mathcal{H}}$-pairing:

Suppose $b \in \mathcal{X}(\mathcal{H})$ and $h \in \mathcal{K}$.

Write $h = \sum f_i g_i$ where $f_i, g_i \in \mathcal{H}$ and

$$\sum \|f_i\|_{\mathcal{H}} \|g_i\|_{\mathcal{H}} \le C \|h\|_{\mathcal{K}}.$$

Then
$$|\langle h | b \rangle| = \left| \left\langle \sum f_i g_i \,\middle|\, b \right\rangle_{\mathcal{H}} \right| \le \sum |\langle f_i g_i | b \rangle_{\mathcal{H}}| \le C \|b\|_{\mathcal{X}(\mathcal{H})} \|h\|_{\mathcal{K}}$$

Conversely suppose $\Lambda \in \mathcal{K}^*$ and $f \in \mathcal{H}$.

Then
$$|\Lambda f| = |\Lambda(f \cdot 1)| \le \|\Lambda\|_{\mathcal{K}^*} \|f \cdot 1\|_{\mathcal{K}} \le \|\Lambda\|_{\mathcal{K}^*} \tilde{C} \|f\|_{\mathcal{H}} \|1\|_{\mathcal{H}}$$

It follows that $\exists!\, b \in \mathcal{H}$ such that

$\Lambda f = \langle f \mid b \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H}$, and it

then follows from $\Lambda \in \mathcal{K}^*$ that

$b \in \mathcal{K}(\mathcal{H})$ and

$$\Lambda\left(\sum f_i s_i\right) = \left\langle \sum f_i s_i \mid b \right\rangle_{\mathcal{H}}$$

for $\sum \|f_i\|_{\mathcal{H}} \|s_i\|_{\mathcal{H}} < \infty$

Now let's turn to

$$\mathcal{D} = \mathcal{D} = \{ f \in \mathrm{Hol}(\mathbb{D}) : \int |f'(z)|^2 \, dA(z) < \infty \}$$

with $\|f\|_{\mathcal{D}}^2 = \|f\|_{H^2}^2 + \int |f'(z)|^2 \frac{dA(z)}{\pi}$

We're interested in the question of which $b \in \mathcal{D}$ define a bounded Hankel form via the formula

$$T_b(f, g) = \langle fg \mid b \rangle_{\mathcal{D}}$$

This question was answered by Arcozzi, Rochberg, Sawyer, Wick 2010:

T_b is a bounded Hankel form on $\mathcal{D} \times \mathcal{D}$ iff

$$|b'(z)|^2 \, dA(z)$$

is a Carleson measure for $\mathcal{D}$

To relate this to our previous discussion we need a space $\mathcal{K}$ which has a weak factorization into $\mathcal{D}$-functions.

We don't have such a space, so we make one up, following ARSW:

$$\mathcal{D} \odot \mathcal{D} = \{h \in \mathrm{Hol}(\mathbb{D}) : h = \sum f_i g_i \text{ with } f_i, g_i \in \mathcal{D} \ \& \ \sum \|f_i\|_{\mathcal{D}} \|g_i\|_{\mathcal{D}} < \infty\}$$

ARSW show that

$$\mathcal{X}(\mathcal{D}) = \{b \in \mathcal{D} : |b'(z)|\, dA(z) \text{ is a Carleson measure for } \mathcal{D}\}$$

and the argument above (which is from ARSW) then shows that this latter space is $(\mathcal{D} \odot \mathcal{D})^*$

Let's examine some of this more closely.

Define

$$\mathcal{D} \hat{\odot} \mathcal{D} = \left\{ h = \sum_{1}^{n} f_i g_i : n \in \mathbb{N},\ f_i, g_i \in \mathcal{D} \text{ for } 1 \le i \le n \right\}$$

with

$$\|h\|_{\hat{*}} = \inf \left\{ \sum_{1}^{n} \|f_i\|_{\mathcal{D}} \|g_i\|_{\mathcal{D}} : h = \sum_{i=1}^{n} f_i g_i \right\}$$

and

$$\mathcal{D} \odot \mathcal{D} = \left\{ h \in \mathrm{Hol}\,\mathbb{D} : h(z) = \sum_{i=1}^{\infty} f_i(z) g_i(z) \text{ where } f_i, g_i \in \mathcal{D} \text{ and } \sum_{i=1}^{\infty} \|f_i\|_{\mathcal{D}} \|g_i\|_{\mathcal{D}} < \infty \right\}$$

with $\|h\|_{*}$ as above with infinite sums

ARSW define $\mathcal{D} \odot \mathcal{D}$ as the completion of $\mathcal{D} \hat{\odot} \mathcal{D}$ but its not obvious that this works.

<u>Example</u> Define $\|\cdot\|$ on polynomials $\mathcal{P}$ by $\|f\|^2 = |f(1)|^2 + \|f\|_{H^2}^2$

It is easy to see that $\exists f_n \in \mathcal{P}$ with $\|f_n\| = 1$, $f_n(1) = 1$ $\forall n$ and $f_n(z) \to 0$ $\forall z \in \mathbb{D}$.

This shows that the completion of $(\mathcal{P}, \|\cdot\|)$ is $\mathbb{C} \oplus H^2$, which is not a space of holomorphic functions.

It is fairly easy to show that (our) $\mathcal{D} \odot \mathcal{D}$ is complete, hence a Banach space of analytic functions, and we would like to show that it can be naturally identified with the completion of $\mathcal{D} \hat{\odot} \mathcal{D}$, i.e. we want to prove the

Theorem [RS]

$$\|h\|_{\hat{*}} = \|h\|_{*} \quad \text{for } h \in \mathcal{D} \hat{\odot} \mathcal{D}$$

Proof Obviously

$$\|h\|_{*} \leq \|h\|_{\hat{*}}$$

For the opposite inequality we need the

Claim

Suppose $\{h_n\}$ is a $\|\cdot\|_{\hat{*}}$ - Cauchy sequence in $\mathcal{D} \hat{\odot} \mathcal{D}$ and $h_n(z) \to 0$ for all $z \in \mathbb{D}$.

Then $\|h_n\|_{\hat{*}} \to 0$

Proof For $0 \leq r < 1$ define

$$\rho_r(f)(z) = f(rz)$$

Then $\|\rho_r(f)\|_{\mathcal{D}} \leq \|f\|_{\mathcal{D}}$ and $\rho_r(f) \to f$ in $\mathcal{D}$ $\forall f \in \mathcal{D}$

It follows that $\|\rho_r(h)\|_{\hat{*}} \leq \|h\|_{\hat{*}}$

& $\rho_r(h) \to h$ in $\|\cdot\|_{\hat{*}}$ as $r \nearrow 1$ $\forall h \in \mathcal{D} \hat{\odot} \mathcal{D}$

and same for $\mathcal{D} \odot \mathcal{D}$

Now $\{h_n\}$ $\|\cdot\|_{\hat{k}}$ – Cauchy implies that $p_r(h_n) \to h_n$ in $\|\cdot\|_{\hat{k}}$ as $r \uparrow 1$ uniformly in n

Now for $\epsilon > 0$ pick $r \in [0,1)$ such that $\|p_r(h_n) - h_n\|_{\hat{k}} < \epsilon$ $\forall n$

Then pick N such that

$$|h_n(z)|, |h_n'(z)| < \epsilon \text{ if } n \geq N \text{ \& } |z| \leq r$$

It follows that $\|p_r(h_n)\|_D < \sqrt{2}\epsilon$ for $n \geq N$,

so $n \geq N \Rightarrow$

$$\begin{aligned} \|h_n\|_{\hat{k}} &\leq \|h_n - p_r(h_n)\|_{\hat{k}} + \|p_r(h_n)\|_{\hat{k}} \\ &\leq \|h_n - p_r(h_n)\|_{\hat{k}} + \|p_r(h_n)\|_D \\ &< (1+\sqrt{2})\epsilon \end{aligned}$$

$\blacksquare$

Continuing with Proof of theorem:

$h \in \mathcal{D} \hat{\otimes} \mathcal{D}$,

$h = \sum_{i=1}^{n} f_i g_i \qquad f_i, g_i \in \mathcal{D}$

Say $h = \sum_{j=1}^{\infty} \tilde{f}_i \tilde{g}_i$ with

$\sum_{j=1}^{\infty} \|\tilde{f}_i\|_{\mathcal{D}} \|\tilde{g}_i\|_{\mathcal{D}} < \|h\|_* + \epsilon$

Let $h_n = h - \sum_{j=1}^{n} \tilde{f}_i \tilde{g}_i = \sum_{j=n+1}^{\infty} \tilde{f}_i \tilde{g}_i$

Apply Claim, get $\|h_n\|_* \to 0$.

So we can find $n, \tilde{\tilde{f}}_i, \tilde{\tilde{g}}_i \in \mathcal{D}$

such that $h_n = \sum_{i=1}^{m} \tilde{\tilde{f}}_i \tilde{\tilde{g}}_i$ and

$\sum_{i=1}^{m} \|\tilde{\tilde{f}}_i\|_{\mathcal{D}} \|\tilde{\tilde{g}}_i\|_{\mathcal{D}} < \epsilon$.

~~Than~~

Then

$$h = h - h_n + h_n$$

$$= \sum_{i=1}^{n} \tilde{f}_i \tilde{s}_i + \sum_{i=1}^{m} \hat{\tilde{f}}_i \hat{\tilde{s}}_i$$

and $$\sum_{i=1}^{n} \|\tilde{f}_i\|_D \|\tilde{s}_i\|_D + \sum_{i=1}^{m} \|\hat{\tilde{f}}_i\|_D \|\hat{\tilde{s}}_i\|_D$$

$$\le \sum_{i=1}^{n} \|\hat{f}_i\|_D \|\hat{s}_i\|_D + \epsilon$$

$$\le \|h\|_* + 2\epsilon \qquad \blacksquare$$

Of course we would really like an "intrinsic definition" of $\mathbb{D} \odot \mathbb{D}$, but ??

Unfortunately we don't know the answer to:

If $h \in \mathbb{D} \odot \mathbb{D}$
do there exist $f, g \in \mathbb{D}$
such that $h = fg$?

In fact we can't even answer:

If $f, g \in \mathbb{D}$ is
$1 + fg$ the product of two
Dirichlet functions?

BUT:

Let $\mathcal{D}_{\mathbb{R}} = \mathrm{Re}\,\mathcal{D}$ = "real Dirichlet space"

Obviously $\mathcal{D}_{\mathbb{R}} \subseteq L^2_{\mathbb{R}}$

so $f, g \in \mathcal{D}_{\mathbb{R}} \Rightarrow fg \in L^1_{\mathbb{R}}$ [in fact $L^2_{\mathbb{R}}$]

We can now define $\mathcal{D}_{\mathbb{R}} \odot \mathcal{D}_{\mathbb{R}} \subseteq L^1_{\mathbb{R}}$ pretty much as above and then prove:

<u>Theorem</u> [RS]

Let $h \in \mathcal{D}_{\mathbb{R}} \odot \mathcal{D}_{\mathbb{R}}$. Then $\exists f, g \in \mathcal{D}_{\mathbb{R}}$ such that $h = fg$

<u>Cor</u> Let $h \in \mathcal{D} \odot \mathcal{D}$. Then $\exists u_1, v_1, u_2, v_2 \in \mathcal{D}_{\mathbb{R}}$ such that

$$h = u_1 u_2 + i v_1 v_2$$

We can relate $\mathbb{D} \odot \mathbb{D}$ and $\mathbb{D}_{\mathbb{R}} \odot \mathbb{D}_{\mathbb{R}}$ in the following nice way.

<u>Theorem</u> [RS] Let $P = P_{\geq 0}$ be the Riesz projection. Then

$$P_{\geq 0}(\mathbb{D}_{\mathbb{R}} \odot \mathbb{D}_{\mathbb{R}}) = \mathbb{D}_{+} \odot \mathbb{D}_{+} \quad (!!)$$

– in contrast to the $H^2 - H^1$ situation

The proof of this result uses the full strength of ARSW.

Let's generalize!

Our proof that $\mathbb{D} \odot \mathbb{D}$ = completion of $\mathbb{D} \hat{\odot} \mathbb{D}$ works for $\mathcal{H}$ a space of analytic functions on Ω works provided that, e.g.

$\Omega = \mathbb{B}_d$ or $\mathbb{D}^d$

$C^\infty(\bar{\Omega}) \subset \mathcal{H}$ continuously

$\|p_r(f)\|_{\mathcal{H}} \leq C\|f\|_{\mathcal{H}}$

$p_r \to \mathrm{id}_{\mathcal{H}}$ in SOT

– in particular all weighted Bergman spaces with radial weights on $\mathbb{D}$

Look at the "Sobolev scale" of analytic functions on $\mathbb{D}$:

$$\mathcal{H}_s = \Big\{ f(z) = \sum_0^\infty a_n z^n \in Hol(\mathbb{D}) \text{ such that } \|f\|^2_{\mathcal{H}_s} = \sum (n+1)^{2s} |a_n|^2 < \infty \Big\}$$

For $s<0$ these are weighted Bergman spaces with "standard weights" so our result applies. But Horowitz does much better:

<u>Theorem</u> [Horowitz 1977?]

For $s<0$, $\mathcal{H}^1_s = \mathcal{H}^2_s \cdot \mathcal{H}^2_s$

(so $\mathcal{H}^2_s \odot \mathcal{H}^2_s = \mathcal{H}^2_s \hat{\odot} \mathcal{H}^2_s = \mathcal{H}^2_s \cdot \mathcal{H}^2_s$

Our result also works for $s > 0$

$[s = \frac{1}{2} \leftrightarrow \mathcal{D}]$ but if $s > \frac{1}{2}$ its not

needed since $\mathcal{H}^2_s$ is then an algebra.

(Real) Sobolev spaces on $\partial\mathbb{D}$:

W_s : $u(e^{i\theta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$

with $\|u\|_s^2 = \sum (|n|+1)^{2s} |a_n|^2 < \infty$

Again for $s > \frac{1}{2}$ W_s an algebra and

for $0 \leq s \leq \frac{1}{2}$ we have that

$$W_s \odot W_s = W_s \hat{\odot} W_s = W_s \cdot W_s$$

For $s < 0$???