Bergman-type Singular Operators and the characterization of Carleson measures for Besov–Sobolev spaces on the complex ball

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The spaces $B^2_\sigma(\mathbb{B}_d)$ is the collection of analytic functions on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ such that for any integer $m \geq 0$ and any $0 \leq \sigma < \infty$ such that $m + \sigma > \frac{d}{2}$ we have the following norm being finite:

$$\|f\|^2_{B^\sigma_2} := \sum_{j=0}^{m-1} |f^{(j)}(0)|^2 + \int_{\mathbb{B}_d} |(1-|z|^2)^{m+\sigma} f^{(m)}(z)|^2 \frac{d\,V(z)}{(1-|z|^2)^{d+1}}.$$

One can show that these spaces are independent of m and are Hilbert spaces, with obvious inner products. The spaces $B^\sigma_2(\mathbb{B}_d)$ are reproducing kernel Hilbert spaces with kernels given by $k^\sigma_\lambda(z) = \frac{1}{(1-\overline{\lambda}\cdot z)^{2\sigma}}$. A minor modification has to be made when $\sigma = 0$, but this introduces a logarithmic reproducing kernel.

A non-negative measure μ supported inside $\mathbb{B}_d$ is called a $B^2_\sigma(\mathbb{B}_d)$-Carleson measure if

$$\int_{\mathbb{B}_d} |f(z)|^2 d\mu(z) \leq C(\mu)^2 \|f\|^2_{B^2_\sigma(\mathbb{B}_d)} \quad \forall f \in B^2_\sigma(\mathbb{B}_d).$$

This is a function theoretic property and is looking for the measures μ that ensure the continuous embedding of $B^2_\sigma(\mathbb{B}_d) \subset L^2(\mathbb{B}_d; \mu)$. There are geometric ways that one can characterize the $B_2^\sigma(\mathbb{B}_d)$-Carleson measures. These characterizations are typically given in terms of the "capacity" associated to the function space $B_2^\sigma(\mathbb{B}_d)$ and an interaction between the geometry of certain sets arising from the reproducing kernel $k_\lambda^\sigma(z)$. Cascante, Ortega, Tchoundja. However, these characterizations had the restriction of only working in the range $0 \leq \sigma \leq \frac{1}{2}$, and when $\frac{d}{2} \leq \sigma$. Namely, previous methods were unable to answer the question in the difficult range of $\frac{1}{2} < \sigma < \frac{d}{2}$. However, using the methods of non-homogeneous harmonic analysis, we can give a characterization of the $B_2^\sigma(\mathbb{B}_d)$ using the Main Theorem in the form written on the next slides for all values of σ at once.

But first we need the following proposition of Arcozzi, Rochberg, Sawyer. Their proposition holds in an arbitrary Hilbert space with a reproducing kernel. Let $\mathcal{J}$ be a Hilbert space of functions on a domain X with reproducing kernel function j_x. In this context, a measure μ is Carleson exactly if the inclusion map ι from $\mathcal{J}$ to $L^2(X;\mu)$ is bounded.

Proposition

A measure μ is a $\mathcal{J}$-Carleson measure if and only if the linear map

$$f(z) \to T(f)(z) = \int_X \operatorname{Re} j_x(z) f(x) d\mu(x)$$

is bounded on $L^2(X;\mu)$.

In $\mathbb{C}^d$, let $\mathbb{B}_d$ denote the open unit ball and consider the kernels given by $K_m(z,w) := \operatorname{Re} \frac{1}{(1-\bar{z}\cdot w)^m}, \quad \forall |z| \le 1, |w| \le 1\,.$

Theorem (Characterization of Carleson Measures for Besov–Sobolev Spaces)

Let μ be a positive Borel measure in $\mathbb{B}_d$. Then the following conditions are equivalent:

(a) *μ is a $B_2^\sigma(\mathbb{B}_d)$-Carleson measure;*

(b) *$T_{\mu,2\sigma} : L^2(\mathbb{B}_d;\mu) \to L^2(\mathbb{B}_d;\mu)$ is bounded;*

(c) *There is a constant C such that*

   (i) $\| T_{\mu,2\sigma}\chi_Q \|^2_{L^2(\mathbb{B}_d;\mu)} \le C\,\mu(Q)$ *for all Δ-cubes Q;*

   (ii) $\mu(B_\Delta(x,r)) \le C\, r^{2\sigma}$ *for all balls $B_\Delta(x,r)$ that intersect $\mathbb{C}^d \setminus \mathbb{B}_d$.*

Let X be a geometrically doubling metric space. Let $\lambda(x,r)$ be a positive function, increasing and doubling in r, i.e. $\lambda(x,2r)\leqslant C\lambda(x,r)$, where C does not depend on x and r. Suppose $K(x,y)\colon X\times X\to\mathbb{R}$ is a Calderon-Zygmund kernel, associated to a function λ, i. e.

$$|K(x,y)|\leqslant C\min\left(\frac{1}{\lambda(x,d(x,y))},\frac{1}{\lambda(y,d(x,y))}\right), \tag{0.1}$$

$$|K(x,y)-K(x',y)|\leqslant C\frac{d(x,x')^{\varepsilon}}{d(x,y)^{\varepsilon}\lambda(x,d(x,y))},\quad d(x,y)\geqslant Cd(x,x'), \tag{0.2}$$

$$|K(x,y)-K(x,y')|\leqslant C\frac{d(y,y')^{\varepsilon}}{|xy|^{\varepsilon}\lambda(y,d(x,y))},\quad d(x,y)\geqslant Cd(y,y'). \tag{0.3}$$

Let μ be a measure on X, such that $\mu(B(x,r))\leqslant C\lambda(x,r)$, where C does not depend on x and r.

We say that T is a Calderon-Zygmund operator with kernel K if

$$T \text{ is bounded } L^2(\mu) \to L^2(\mu), \tag{0.4}$$

$$Tf(x) = \int K(x,y) f(y) d\mu(y), \ \forall x \notin \text{supp}\mu, \ \forall f \in C_0(X). \tag{0.5}$$

Theorem (non-homogeneous T1)

Then testing T, T^* on χ_Q is necessary and sufficient for $L^2(X,\mu)$ boundedness of T.

Our case of $\lambda(x,r)$: let all "non-Ahlfors balls", that is $B(x,r)$ such that $\mu(B(x,r)) > r^m$, lie in an open set $H \subset X$. Let $\lambda(x,r) := \max(\mathrm{dist}(x, X \setminus H), r)^m$. Then the abovementioned relation between λ and μ is satisfied. **Equivalently** kernel k is the usual Calderón–Zygmund kernel with parameters (m,ε) (that is $d(x,y)^m$, $d(x,y)^{m+\varepsilon}$ in the corresponding denominators) that satisfies an extra inequality
$|k(x,y)| \le \frac{1}{\max(\mathrm{dist}(x,X\setminus H), \mathrm{dist}(y,X\setminus H))} =: \frac{1}{\max(d(x),d(y))}$.

Metric on the ball.

We introduce the above mentioned (quasi)-metric on the spherical layer around $\partial\mathbb{B}_d$:

$$\Delta(z,w) := ||z|-|w||+\left|1-\frac{z}{|z|}\frac{w}{|w|}\right|, \, 1/2 \le |z| \le 2\,, \, 1/2 \le |w| \le 2\,.$$

Then it is easy to see that for all $z, w : |z| \le 1, |w| \le 1$, we have

$$|K_m(z,w)| \lesssim \frac{1}{\Delta(z,w)^m}.$$

This holds because we know that $\left|\frac{1}{(1-\bar{z}\cdot w)^m}\right| \lesssim \frac{1}{\Delta(z,w)^m}$ Tchoundja proved that if $\Delta(\zeta,w) << \Delta(z,w)$ then

$$|K_m(\zeta,w)-K_m(z,w)| \le C\frac{\Delta(\zeta,w)^{1/2}}{\Delta(z,w)^{m+1/2}}\,.$$

This estimate then says that the kernel K_m is a Calderón–Zygmund kernel defined on the closed unit ball, but with respect to the quasi metric $\Delta(z,w)$ with associated Calderón–Zygmund parameter $\tau = 1/2$.

Let μ be a probability measure with compact support contained in the spherical layer $1/2 \leq |z| < 1$ and in particular the support is strictly inside the ball. We can see that this kernel satisfies the hypotheses of non-homogeneous $T1$ theorem above when $H = \mathbb{B}_d$, but with respect to a certain (non-euclidean) quasi-metric Δ. It is clear that $d(z) := \text{dist}_\Delta(z, \mathbb{C}^d \setminus \mathbb{B}_d) = 1 - |z|$. Since if $z, w \in \mathbb{B}_d$ we have $|1 - z \cdot w|^m \geq (1 - |z|)^m$ and a similar statement holding for w. Therefore, $|K_m(z, w)| \leq \frac{1}{\max(d(z), d(w))^m}$.

Theorem

Let μ be a probability measure supported in $\{z \in \mathbb{C}^d : 1/2 \le |z| < 1\}$. Then the following assertions are equivalent:

(i) $\mu(B_\Delta(x,r)) \le C_1\, r^m\,,\ \forall B_\Delta(x,r) : B_\Delta(x,r) \cap \mathbb{C}^d \setminus \mathbb{B}_d \neq \emptyset;$

(ii) *For all Δ-cubes Q we have $\| T_{\mu,m}\chi_Q\|^2_{L^2(X;\mu)} \le C_2\mu(Q)$.*

and

$$\| T_{\mu,m} : L^2(X;\mu) \to L^2(X;\mu)\| \le C_3 < \infty\,.$$

Here $C_3 = C(C_1, C_2, m)$, $C_1 = C(C_3)$, $C_2 = C(C_3)$.

Theorem (Complex version)

Let $k(z, w)$ be a Calderón–Zygmund kernel of order m on $X := \{1/2 \le |z| \le 2\} \subset \mathbb{C}^d$, $m \le 2d$ with Calderón–Zygmund constants C_{CZ} and τ, but with respect to the metric Δ introduced above. Let μ be a probability measure with compact support in $X \cap \mathbb{B}_d$, and suppose that all balls B_Δ in the metric Δ such that $\mu(B_\Delta(x, r)) > r^m$ lie in an open set H. Let also

$$|k(z, w)| \le \frac{1}{\max(d(z)^m, d(w)^m)} \,,$$

where $d(z) := \mathrm{dist}_\Delta(z, \mathbb{C}^d \setminus H)$. Finally, suppose also that a "T1 Condition" holds for the operator T with kernel k and for the operator T^* with kernel $k(w, z)$:

$$\|T_{\mu,m}\chi_Q\|^2_{L^2(X;\mu)} \le A\,\mu(Q)\,,\ \|T^*_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \le A\,\mu(Q)\,. \quad (0.6)$$

Then $\|T_{\mu,m}\|_{L^2(X;\mu)\to L^2(X:\mu)} \le C(A, m, d, \tau)$.

Theorem (Real version)

Let $k(x,y)$ be a Calderón–Zygmund kernel of order m on $X \subset \mathbb{R}^d$, $m \le d$ with Calderón–Zygmund constants C_{CZ} and τ. Let μ be a probability measure with compact support in X and all balls such that $\mu(B(x,r)) > r^m$ lie in an open set H. Let also

$$|k(x,y)| \le \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \mathrm{dist}(x, \mathbb{R}^d \setminus H)$. Finally, suppose also that a "T1 Condition" holds for the operator $T_{\mu,m}$ with kernel k and for the operator $T^*_{\mu,m}$ with kernel $k(y,x)$:

$$\|T_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \le A\,\mu(Q)\,,\ \|T^*_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \le A\,\mu(Q)\,. \quad (0.7)$$

Then $\|T_{\mu,m}\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \le C(A,m,d,\tau)$.

So our goal now is to forget about any specific setting and to give the proof of non-homogeneous $T1$ theorem on metric spaces with operator T being a Calderón–Zygmund operator in the sense of slides 6 and 7. That is with this λ in denominator.
The novelty of this talk is a new proof of non-homogeneous T1 theorem. Even if $\lambda(x,r) = r^m, d(x,y) = |x-y|$, (euclidean metric space $\mathbb{R}^d$ and the usual Calderón–Zygmund kernel of order $m \le d$) this proof is new and "interesting". But it works without **any** change for any metric space and any (λ, μ) as on slides 6. 7.
Non-homogeneous T1 theorems were first proved by Nazarov–Treil–Volberg (NTV) and by Tolsa. These theorems, and their analogs like various (especially non-accretive) Tb theorems were widely used during the next decade to answer Denjoy's question about Analytic capacity/Geometric Measure Theory : this was done by Mattila, Melnikov, Verdera in "homogeneous" case and by David, Léger, and by NTV in the general case. Also questions of Painlevé, Vitushkin and Ahlfors about Analytic capacity were answered. This was done by Tolsa, he used non-homogeneous non-accretive Tb theorem of NTV.

We are in a position to formulate our main results. Recall when operator T is called an operator with Calderón–Zygmundkernel of order m.
Let X be a geometrically doubling metric space.
Let $\lambda(x, r)$ be a positive function, increasing and doubling in r, i.e. $\lambda(x, 2r) \leqslant C\lambda(x, r)$, where C does not depend on x and r.
Let $K(x, y), \lambda(x, r), \mu$ be as before on slides 6, 7. For simplicity we formulate and prove only the simplest setting of λ and metric.

Theorem

Let $\mu(B(x,r)) \le r^m$. *Let* T *be a Calderón–Zygmund operator of order* m *in* $\mathbb{R}^d$. *Then there exists a probability space of dyadic lattices* $(\Omega, \mathcal{P})$ *such that*

$$\begin{aligned} T = c_{1,T} \int_\Omega \Pi(\omega)\, d\mathcal{P}(\omega) + c_{2,T} \int_\Omega \Pi^*(\omega)\, d\mathcal{P}(\omega) + \\ c_{3,T} \sum_{n=0}^{\infty} 2^{-n\varepsilon_T} \int_\Omega \mathbb{S}_n(\omega)\, d\mathcal{P}(\omega)\,. \end{aligned} \tag{0.8}$$

Moreover, $\varepsilon_T > 0$. *Constants* $c_{1,T}, c_{2,T}, c_{3,T}$ *depend on the Calderón–Zygmundparameters of the kernel, on* m *and* d, *and on the best constant in the so-called* $T1$ *conditions:*

$$\|T 1_Q\|_{2,\mu}^2 \le C_0 \mu(Q)\,, \tag{0.9}$$

$$\|T^* 1_Q\|_{2,\mu}^2 \le C_0 \mu(Q)\,, \tag{0.10}$$

The same thing holds on general geometrically doubling metric space X (not just $\mathbb{R}^d$) and any non-homogeneous Calderón–Zygmund operator having Calderón–Zygmund kernel in the generalized sense above. Of course measure should satisfy

$$\mu(B(x,r)) \leqslant C\lambda(x,r)\,.$$

We prefer to prove the $\mathbb{R}^d$-version just for the sake of avoiding some slight technicalities. For example, the construction of the suitable probability space of random dyadic lattice on X is a bit more involved than such construction in $\mathbb{R}^d$. See two different constructions of suitable probability spaces of dyadic lattices in Hytönen–Martikainen and Nazarov–Reznikov–Volberg.

The $T1$ theorem is a corollary of course. It has a long story: if $\mu = m_d$ it was proved by David–Journé. For homogeneous (doubling) measures μ it was proved by Christ. In the case of non-homogeneous μ, T1 theorem was proved in NTV. Just NTV is not quite enough however to prove the above decomposition to shifts, and we use a beautiful step of Hytönen as well. Then non-homogeneous $T1$ theorem is just a corollary of the decomposition result, because all shifts of order n involved in (0.8) have norms at most $n+1$ (see the discussion above), but decomposition (0.8) has an exponentially decreasing factor.

Probability space of dyadic lattices and dyadic shifts of different order

Let $(\Omega, \mathcal{P})$ be a probability space of "dyadic" lattices on our metric space X (below $X = \mathbb{R}^d$, see Nazarov–Reznikov–V. for general X) satisfying certain axioms. In what follows all $\mathcal{D} = \mathcal{D}(\omega) = \cup_{k \le N} \mathcal{D}_k$ are from $(\Omega, \mathcal{P})$. So let Q be in such a $\mathcal{D}$ and let Q_i, $i = 1, \dots, 2^d$ be its children. For any $f \in L^1(\mu)$ we denote $\mathcal{E}_Q f = \langle f \rangle_{1,\mu} 1_Q$,

$$\mathcal{E}_k := \sum_{Q \in \mathcal{D}_k} \mathcal{E}_Q f \,,$$

and

$$\Delta_k f = (\mathcal{E}_{k+1} - \mathcal{E}_k) f, \; \Delta_Q f := \Delta_k f \cdot 1_Q \,, Q \in \mathcal{D}_k \,.$$

Now let $f \in L^2_0(\mu)$ subscript 0 meaning that $\int f\, d\mu = 0$. Then

$$f = \sum_{Q \in \mathcal{D}} \Delta_Q f\,,$$

and

$$\Delta_Q f = \sum_{i=1}^{2^d-1} (f, h^i_Q)_\mu h^i_Q\,,$$

where h^i_Q are called ($\mu-$ Haar functions and the have the following properties

- $(h^i_Q, h^j_R)_\mu = 0, Q \neq R,$
- $(h^i_Q, h^j_Q)_\mu = 0, i \neq j,$
- $\|h^i_Q\|_\mu = 1,$
- $h^i_Q = \sum_{m=1}^{2^d-1} c^i_{Q,m} 1_{Q_m},$
- $|c^i_{Q,m}| \leq 1/\sqrt{\mu(Q_m)}.$

Above Q_m's are children of Q. Below we will start to skip index i. We already abbreviated $h^{i,\mu}_Q$ to h^i_Q now we abbreviate further to h_Q unless said otherwise.

Definition of good cubes

Definition

Cube $Q \in \mathcal{D}(\omega)$ is called **good** *((r,γ)-good) if for any R in the same $\mathcal{D}(\omega)$ but such that $\ell(R) \geq 2^r \ell(Q)$ one has*

$$\operatorname{dist}(Q, sk(R)) \geq \ell(R)^{1-\gamma}\ell(Q)^{\gamma}\,, \tag{0.11}$$

where $sk(R) := \bigcup_{m=1}^{2^d-1} \partial Q_m$, again Q_m's being all children of Q.

Given $Q \in \mathcal{D}_k$ we denote $g(Q) = k$. Our main "tool" is going to be "dyadic shifts". But they will be with respect to non-homogenous measure. Their typical building blocks will be Haar projections with respect to non-homogeneous measure μ.

Axiom of $(\Omega, \mathcal{P})$. $\forall Q, \ \ \mathcal{P}\{Q \text{ is bad}\} = \frac{1}{2}$.

Dyadic shifts

Definition

We call by $\mathbb{S}_{m,n}$ (*shift of complexity* (m,n), *or shift of complexity* $\max(m,n)$) *the operator given by the kernel*

$$f \to \sum_{L\in\mathcal{D}} \int_L a_L(x,y) f(y)\, d\mu(y)\,, \ \ \textit{where}$$

$$a_L(x,y) = \sum_{\substack{Q\subset L, R\subset L\\ g(Q)=g(L)+m,\, g(R)=g(L)+n}} c_{L,Q,R} h^i_Q(x) h^j_R(y)\,, \tag{0.12}$$

where $h^i_Q := h^{\mu,i}_Q$, $h^j_R := h^{\mu,j}_R$ *are Haar functions (as above) orthogonal and normalized in* $L^2(d\mu)$, *and* $|c_{L,Q,R}|$ *are such that*

$$\sum_{Q,R} |c_{L,Q,R}|^2 \le 1\,. \tag{0.13}$$

Remark

In particular, it is easy to see that if a_L has form (0.12) and satisfies

$$|a_L(x,y)| \leq \frac{1}{\mu(L)}\,, \tag{0.14}$$

then (0.13) is automatically satisfied, and we are dealing with dyadic shift.
A little bit different but basically equivalent definition can be given like that: operator sending $\Delta_L^n(L^2(\mu))$ to itself and having the kernel $a_L(x,y)$ satisfying estimate (0.14) is called a local dyadic shift of order n. Here $\Delta_L^n(L^2(\mu))$ denotes the space of $L^2(\mu)$ functions supported on L and having constant values on children Q of L such that $g(Q) = g(L) + n + 1$. Now dyadic shift of order n is an operator of the form $\mathbb{S}_n f := \sum_{L\in\mathcal{D}} \int_L a_L(x,y) f(y) dy$, where a_L corresponds to local shift of order n.

All these definitions bring us operators satisfying obviously $\|\mathbb{S}_{m,n}\|_{L^2(\mu)\to L^2(\mu)} \leq 1$, or $\|\mathbb{S}_n\|_{L^2(\mu)\to L^2(\mu)} \leq n+1$.

Generalized shifts

We also need generalized shifts, but only of complexity $(0, 1)$.

Definition

Let $\Pi f := \sum_{L\in\mathcal{D}} \langle f \rangle_{L,\mu} \sqrt{\mu(L)} \sum_{\ell\subset L, |\ell|=2^{-s}|L|} c_{L,\ell} \cdot h^j_\ell$, *where* $\{c_{L,\ell}\}$ *satisfy not just the condition* $\mu(L) \sum_{\ell\subset L, |\ell|=2^{-s}|L|} |c_{\ell,L}|^2 \le 1$ *that would be equivalent to "the usual* $(0, s)$*-shift normalization condition", but a rather stronger Carleson condition*

$$\sum_{L\subset R,\, L\in\mathcal{D}} \mu(L) \sum_{\ell\subset L, |\ell|=2^{-s}|L|} |c_{\ell,L}|^2 \le \mu(R)\,. \tag{0.15}$$

Then Π *is called a generalized shift of complexity* $(0, s)$.

With $s = 1$ these are paraproducts. We will need only paraproducts and their duals.

Proof of non-homogeneous decomposition theorem

We are proving now the decomposition to shifts Theorem of slide 15, which immediately gives non-homogeneous $T1$ theorem. Let $f, g \in L^2_0(\mu)$, having constant value on each cube from $\mathcal{D}_N$. We can write

$$f = \sum_Q \sum_j (f, h^j_Q) h^j_Q, \qquad g = \sum_R \sum_i (g, h^i_R) h^i_R.$$

First, we state and proof the theorem, that says that essential part of bilinear form $(Tf, g)_\mu$ of T can be expressed in terms of pair of cubes, where the smallest one is good. This is almost what has been done in NTV 1997, 2002. The difference is that in NTV an error term (very small) appeared. To eliminate the error term we follow the idea of Hytönen. In fact, the work Hytönen improved on "good-bad" decomposition of NTV by replacing inequalities by an equality and getting rid of the error term.

Representation by the sum, where the smaller in size is always good

Theorem

Let T be any linear operator. Then the following equality holds:

$$\frac{1}{2}\mathcal{E}\sum_{\substack{Q,R,i,j\\ \ell(Q)\geqslant\ell(R)}}(Th_Q^j,h_R^i)(f,h_Q^j)(g,h_R^i)=$$

$$\mathcal{E}\sum_{\substack{Q,R,i,j\\ \ell(Q)\geqslant\ell(R),\ R\text{ is good}}}(Th_Q^j,h_R^i)(f,h_Q^j)(g,h_R^i).$$

The same is true if we replace $\geqslant$ by $>$.

Proof

We denote $\sigma(T) = \sum_{\ell(Q)\geqslant\ell(R)} (Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R)$,
$\sigma'(T) = \sum_{\substack{\ell(Q)\geqslant\ell(R)\\ R \text{ is good}}} (Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R)$. We would like to get a relationship between $\mathcal{E}\,\sigma(T)$ and $\mathcal{E}\,\sigma'(T)$. We fix R and put $g_{good} := \sum_{R \text{ is good}} (g, h^i_R)h^i_R$. Then

$$\sum_Q \sum_{R \text{ is good}} (Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R) =$$

$$\left(T(f), \sum_{R \text{ is good}} (g, h^i_R)h^i_R\right) = (T(f), g_{good})\,.$$

Take expectations

$$\mathcal{E}\sum_{Q,R}(Th^j_Q,h^i_R)(f,h^j_Q)(g,h^i_R)\mathbf{1}_{R\,\text{is good}}=\mathcal{E}(T(f),g_{good})=$$

$$(T(f),\mathcal{E}\,g_{good})=\frac{1}{2}(T(f),g)=\frac{1}{2}\mathcal{E}\sum_{Q,R}(Th^j_Q,h^i_R)(f,h^j_Q)(g,h^i_R). \tag{0.16}$$

Fine. But we need "triangle" sum: where $\ell(R)\le\ell(Q)$! Fix a pair $\ell(Q)<\ell(R)$. Then the goodness of R does not depend on Q, so

$$\frac{1}{2}(Th^j_Q,h^i_R)(f,h^j_Q)(g,h^i_R)=\mathcal{E}\left((Th^j_Q,h^i_R)(f,h^j_Q)(g,h^i_R)\mathbf{1}_{R\,\text{is good}}|Q,R\right)$$

Let us explain this equality. The right hand side is conditioned: meaning that the left hand side involves the fraction of two numbers: 1) the number of all lattices containing Q,R in it and such that R (the one that is larger by size) is good and 2) the number of lattices containing Q,R in it. This fraction is exactly $\pi_{good}=\frac{1}{2}$. The equality has been explained.

$$\frac{1}{2}\mathcal{E}\sum_{Q,R}(Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R) = \mathcal{E}\sum_{Q,R}(Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R)\mathbf{1}_{R\ \text{good}} =$$

$$\mathcal{E}\sum_{\ell(Q)<\ell(R)}(Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R)\mathbf{1}_{R\,\text{good}} + \mathcal{E}\sum_{\ell(Q)\geqslant\ell(R)}(Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R$$

$$\frac{1}{2}\mathcal{E}\sum_{\ell(Q)<\ell(R)}(Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R) + \mathcal{E}\sum_{\ell(Q)\geqslant\ell(R),R\ \text{good}}(Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R),$$

and therefore

$$\mathcal{E}\sum_{\ell(Q)\geqslant\ell(R),\ R\ \ \text{good}}(Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R) = \frac{1}{2}\mathcal{E}\sum_{\ell(Q)\geqslant\ell(R)}(Th^j_Q, h^i_R)(f, h^j_Q)(g, h^i_R) \tag{0.17}$$

which is the statement we wanted.

We have just reduced the estimate of the bilinear form $\sum_{Q,R\in\mathcal{D}}(Th_Q, h_R)(f, h_Q)(g, h_R)$ to the estimate over *all* dyadic lattices in our family, but summing over pairs Q, R, where the smaller in size is always *good*:
$\mathcal{E}\sum_{Q,R\in\mathcal{D},\,\text{smaller is good}}(Th_Q, h_R)(f, h_Q)$. Split it to two "triangular" sums:

$$\mathcal{E}\sum_{Q,R\in\mathcal{D},\,\ell(R)<\ell(Q),\,R\text{ is good}}(Th_Q, h_R)(f, h_Q)(g, h_R)$$

and

$$\mathcal{E}\sum_{Q,R\in\mathcal{D},\,\ell(Q)\le\ell(R),\,Q\text{ is good}}(Th_Q, h_R)(f, h_Q)(g, h_R)\,.$$

They are symmetric, so we will work only with the second sum.

First consider $\sigma_0 := \mathcal{E} \sum_{Q\in\mathcal{D},\ Q \text{ is good}} (Th_Q, h_Q)(f, h_Q)(g, h_Q)$. We do not care where Q is good or not and estimate the coefficient (Th_Q, h_Q) in the most simple way. Recall that $h_Q = \sum_{j=1}^{2^d} c_{Q,j} 1_{Q_j}$, where Q_j are children of Q. We also remember that $|c_{Q,j}| \le 1/\sqrt{\mu(Q_j)}$. Estimating

$$|c_{Q,j}||c_{Q,j'}||(T1_{Q_j}, 1_{Q_{j'}})| \le 1/\sqrt{\mu(Q_j)} 1/\sqrt{\mu(Q_j)} C_0^2 \sqrt{\mu(Q_j)} \sqrt{\mu(Q_j)} \le C_0^2$$

by (0.9), we can conclude that σ_0 / C_0^2 is actually splits to at most 4^d shifts of order 0. Similarly we can work with $s = 1, \dots, r$. We need r to be large but fixed.

$$\sigma_s := \mathcal{E} \sum_{Q,R\in\mathcal{D},\, Q\subset R,\, \ell(Q)=2^{-s}\ell(R),\, Q \text{ is good}} (Th_Q, h_R)(f, h_Q)(g, h_R)\,. \qquad (0.18)$$

Inner sum: $Q \subset R$, Q is good

We gather good Q and some R like above and also $\ell(Q) \le 2^{-r}\ell(R)$. Then it is easy to see that $\mathrm{dist}(Q, \partial R_2) \le 2^{-r-1}\ell(R_1) = \ell(Q)$, where R_1 is a descendant of R such that $\ell(R_1) = 2^r\ell(Q)$, and R_2 is son of R_1 that contains Q.

Lemma

Let $Q \subset R$, $S(R)$ be the son of R containing Q, and let $\mathrm{dist}(Q, \partial S(R)) \ge \ell(Q)$. Let T be a Calderón–Zygmund operator with parameter ε. Then
$(Th_Q, h_R) = \langle h_R\rangle_{S(R)}(h_Q, \Delta_Q T^*1)_\mu + t_{Q,R}$, *where*

$$|t_{Q,R}| \le \int_Q \int_{R\setminus S(R)} \frac{\ell(Q)^\varepsilon}{\mathrm{dist}(t, Q) + \ell(Q))^{m+\varepsilon}} |h_Q(s)||h_R(t)|\, d\mu(s)|\, d\mu(t)$$

$$\int_Q \int_{\mathbb{R}^d\setminus S(R)} \frac{\ell(Q)^\varepsilon}{\mathrm{dist}(t, Q) + \ell(Q))^{m+\varepsilon}} |h_Q(s)||\langle h_R\rangle_{S(R),\mu}|\, d\mu(s)|\, d\mu(t)\,.$$

Consider two integral terms above separately
$t_1 := t_{1,Q,R} := \int_{R\setminus S(R)} \dots$ and $t_2 := t_{2,Q,R} := \int_{\mathbb{R}^d\setminus R} \dots$.
In the second integral we estimate h_Q in $L^1(\mu)$:
$\|h_Q\|_{1,\mu} \le \sqrt{\mu(Q)}$, and we estimate h_R in $L^\infty(\mu)$:
$\|h_R\|_\infty \le 1/\sqrt{\mu(S(R))}$. Integral itself is at most (recall that $\mu(B(x,r) \le r^m))$

$$\int_{\mathbb{R}^d\setminus S(R)} \frac{\ell(Q)^\varepsilon}{(\mathrm{dist}(t,Q)+\ell(Q))^{m+\varepsilon}}\,d\mu(t) \le \frac{\ell(Q)^\varepsilon}{\mathrm{dist}(Q,sk(R))^\varepsilon}\,. \tag{0.19}$$

So if Q is good, meaning that $\mathrm{dist}(Q,sk(R)) \ge \ell(R)^{1-\gamma}\ell(Q)^\gamma$ then (0.19) gives us

$$|t_{2,Q,R}| \le \left(\frac{\mu(Q)}{\mu(S(R))}\right)^{1/2}\frac{\ell(Q)^{1-\varepsilon\gamma}}{\ell(R)^{1-\varepsilon\gamma}}\,. \tag{0.20}$$

In the first integral we estimate h_Q in $L^1(\mu)$: $\|h_Q\|_{1,\mu} \le \sqrt{\mu(Q)}$, and we *cannot* estimate h_R in $L^\infty(\mu)$: $\|h_R\|_{L^\infty(R\setminus S(R))}$. The problem is that this supremum is bounded by $1/\sqrt{\mu(s(R))}$ for a sibling $s(R)$ of $S(R)$. But because doubling is missing this can be an uncontrollably bad estimate. The term $\left(\frac{\mu(Q)}{\mu(S(R))}\right)^{1/2}$ is a good term , at least it is bounded by 1, on the other hand the term $\left(\frac{\mu(Q)}{\mu(s(R))}\right)^{1/2}$ is not bounded by anything, it is uncontrollable. Therefore, we estimate here $\|h_R\|_{1,\mu} \le \sqrt{\mu(R)}$. THUS, we are forced to estimate Integral kernel itself in L^∞ as all $L^1(\mu)$ has been just spent. So we get the term $\frac{\ell(Q)^\varepsilon}{\mathrm{dist}(Q, sk(R))^{m+\varepsilon}}$.

As Q is **good**, meaning that $\mathrm{dist}(Q, sk(R)) \ge \ell(R)^{1-\gamma}\ell(Q)^\gamma$ then we got $|t_{1,Q,R}| \le (\mu(Q)\mu(R))^{1/2} \frac{\ell(Q)^\epsilon}{\ell(R)^{m+\varepsilon-(m+\varepsilon)\gamma}\ell(Q)^{(m+\varepsilon)\gamma}}$. Choose $\gamma := \frac{\varepsilon}{2(m+\varepsilon)}$. Then

$$|t_{1,Q,R}| \le \left(\frac{\ell(Q)}{\ell(R)}\right)^{\varepsilon/2} \frac{\sqrt{\mu(Q)}\sqrt{\mu(R)}}{\ell(R)^m} \le \left(\frac{\mu(Q)}{\mu(R)}\right)^{1/2} \left(\frac{\ell(Q)}{\ell(R)}\right)^{\varepsilon/2}.$$

Inner sums $\sum_{Q\subset R,\, \ell(Q)=2^{-r-k}\ell(R),\, Q \text{ is good}} (Th_Q, h_R)(f, h_Q), (g, h_R)$, $k \geq 0$, can be written as three sums:

$$\sum_R \sum_{Q\subset R,\, \ell(Q)=2^{-r-k}\ell(R),\, Q \text{ is good}} t_{1,Q,R}(f, h_Q), (g, h_R)\,,$$

$$\sum_R \sum_{Q\subset R,\, \ell(Q)=2^{-r-k}\ell(R),\, Q \text{ is good}} t_{2,Q,R}(f, h_Q), (g, h_R)\,,$$

and

$$\sum_R \sum_{Q\subset R,\, \ell(Q)=2^{-r-k}\ell(R),\, Q \text{ is good}} \langle h_R\rangle_{S(R)} (h_Q, \Delta_Q T^*1)_\mu (f, h_Q)(g, h_R)\,.$$

Obviously, the first sum is the bilinear form of a shift of complexity $(0, r+k)$ having the coefficient $2^{-\frac{\varepsilon(r+k)}{2}}$ in front. The second sum is also the bilinear form of a shift of complexity $(0, r+k)$ having the coefficient $2^{-\frac{(1-\varepsilon\gamma)(r+k)}{2}}$ in front. We just look at two . previous slides and notice that $\sum_{Q\subset S(R)} \left((\mu(Q)/\mu(S(R))^{1/2} \right)^2 \leq 1$.

The sum of third sums is a paraproduct

The outer sum. Why it is the combination of shifts?

The decomposition of the outer sum We are left to decompose

$$\mathcal{E} \sum_{Q\cap R=\emptyset,\, \ell(Q)\le\ell(R),\, Q \text{ is good}} (Th_Q, h_R)(f, h_Q, g, h_R)$$

into the bilinear form of (s, t)-shifts with exponentially small in $\max(s, t)$ coefficients.
Denote

$$D(Q, R) := \ell(Q) + \mathrm{dist}(Q, R) + \ell(R)\,.$$

Also let $L(Q, R)$ be a dyadic interval from the same lattice such that $\ell(L(Q, R) \in 2D(Q, R), 4D(Q, R))$ that contains R.

Exactly as we did this before we can estimate

$$(Th_Q, h_R) = \int_R \int_Q [K(x,y) - K(x, c_Q)] h_Q(y) h_R(x)\, d\mu(y)\, d\mu(x)$$

by estimating $\|h_Q\|_{1,\mu} \le \sqrt{\mu(Q)}$, $\|h_R\|_{1,\mu} \le \sqrt{\mu(R)}$, and $\frac{\ell(Q)\varepsilon}{\operatorname{dist}(Q,R)^{m+\varepsilon}} \le \ell(Q)^{\varepsilon/2}/\ell(R)^{\varepsilon/2+m}$ if $\operatorname{dist}(Q,R) \le \ell(R)$. Otherwise the estimate is $\ell(Q)^{\varepsilon}/D(Q,R)^{m+\varepsilon}$. These two estimates are both united into the following one obviously

$$|(Th_Q, h_R)| \le C\Big(\frac{\ell(Q)}{\ell(R)}\Big)^{\varepsilon/2} \frac{\ell(R)^{\varepsilon/2}}{D(Q,R)^{m+\varepsilon/2}} \sqrt{\mu(Q)}\sqrt{\mu(R)}\,. \quad (0.21)$$

Of course in this estimate we used not only that Q is good, but also that $\ell(Q) \le 2^{-r}\ell(R)$. Only having this latter condition we can apply the estimate on $\operatorname{dist}(Q,R)$ that was used in getting the previous inequality (0.21).

Simple pairs: comparable size

However, if $\ell(Q) \in [2^{-r-1}\ell(R), \ell(R)]$ we use just a trivial estimate of coefficient $(Th_Q, h_R)|$, namely

$$|(Th_Q, h_R)| \le C(C_0, d)\,, \tag{0.22}$$

where C_0 is from (0.9). This is not dangerous at all because such pairs Q, R will be able to form below only shifts of complexity (s, t), where $0 \le s \le t \le r$; the number of such shifts is at most $\frac{r(r+1)}{2}$, and let us recall, that r is not a large number, it depends only on d.

Now in a given $\mathcal{D} \in \Omega$ a pair of Q, R may or may not be inside $L(Q, R)$ ($R \subset L(Q, R)$ by definition). But the ratio of **nice lattices** (these are those when both Q, R are inside $L(Q, R)$) with respect to all lattices in which both Q, R are present is bounded away from zero, this ratio (probability) satisfies

$$p(Q, R) \geq P_d > 0\,. \tag{0.23}$$

We want to modify the following expectation

$$\Sigma := \mathcal{E} \sum_{Q \cap R = \emptyset,\, \ell(Q) \leq \ell(R),\, Q \text{ is good}} (Th_Q, h_R)(f, h_Q, g, h_R)\,.$$

This expectation is really a certain sum itself, namely the sum over all lattices in Ω divided by their total number $\sharp(\Omega) =: M$. Each time Q, R are not in a nice lattice we put zero in front of corresponding term. This changes very much the sum. However we can make up for that, and we can **leave the sum unchanged** if for nice lattices we put the coefficient $1/p(Q, R)$ in front of corresponding terms (and keep 0 otherwise).

Then

$$\frac{\text{number of lattices containing } Q, R}{M} = \frac{1}{p(Q,R)} \frac{\text{number of nice lattices cor}}{M}$$

Notice that in the original sum Σ terms Q, R are multiplied by the LHS. The modified sum will contain the same terms multiplied by the RHS. **So it is not modified at all, it is exactly the same sum!** We can write it again as

$$\mathcal{E} \sum_{Q\cap R=\emptyset,\, \ell(Q)\le\ell(R),\, Q \text{ is good}} m(Q,R,\omega)(Th_Q, h_R)(f, h_Q)(g, h_R)\,,$$

where the random coefficients $m(Q, R, omega)$ are either 0 (if the lattice $\mathcal{D} = \omega$ is not nice), or $1/p(Q, R)$ if the lattice is nice. Now let us fix two positive integers $s \le t$, fix a lattice, and consider this latter sum only for this lattice, and write it as

$$\sum_s \sum_t \sum_L \sum_{Q\subset L, R\subset L, \ell(Q)=2^{-t}\ell(L), \ell(R)=2^{-s}\ell(L)} m(P,Q)(Th_Q, h_R)(f, h_Q)(g, h_R)$$

$$\sum_s \sum_t \sigma_{s,t}\,.$$

Each $\sigma_{s,t}$ is a dyadic shift of complexity (s, t). In fact, use $p(Q, R) \geq P_d > 0$ and use already proved (0.21), which is the following:

$$|(Th_Q, h_R)| \leq C\left(\frac{\ell(Q)}{\ell(R)}\right)^{\varepsilon/2} \frac{\ell(R)^{\varepsilon/2}}{D(Q, R)^{m+\varepsilon/2}} \sqrt{\mu(Q)}\sqrt{\mu(R)}\,.$$

Then one can easily see that the sum of squares of coefficients inside each L is bounded. Moreover, the terms $\left(\frac{\ell(Q)}{\ell(R)}\right)^{\varepsilon/2}$, $\left(\frac{\ell(R)}{D(Q,R)}\right)^{\varepsilon/2}$ from (0.21) gives us the desired exponentially small coefficient whose size is at most
$2^{-\varepsilon(t-s)/2} \cdot 2^{-\varepsilon s/2} = 2^{-\varepsilon t/2} = 2^{-\varepsilon \max(s,t)/2}$.

By T1 operator we understand the operator satisfying the test conditions (along with its adjoint operator).

The theorem on decomposition of T1 operator with Calderón–Zygmund kernel to "dyadic shifts" of various order with exponential (in order) coefficients is completely proved.