Composition of Haar Paraproducts

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Hilbert Function Spaces
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This talk is based on joint work with:

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Eric T. Sawyer

McMaster University

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Sandra Pott

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Maria Reguera Rodriguez

Universidad Autónoma de Barcelona

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The composition of $T_\varphi T_{\overline{\psi}}$ is bounded on $H^2(\mathbb{D})$ if and only if

$$\sup_{z\in\mathbb{D}} \left(\int_{\mathbb{T}} \frac{1-|z|^2}{\left|1-z\overline{\xi}\right|^2} |\varphi(\xi)|^2 \, dm(\xi) \right) \left(\int_{\mathbb{T}} \frac{1-|z|^2}{\left|1-z\overline{\xi}\right|^2} |\psi(\xi)|^2 \, dm(\xi) \right) < \infty$$

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Unfortunately, this is not true! A counterexample was constructed by Nazarov.

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Possible to rephrase this question as one about the two-weight boundedness of the Hilbert transform. Deep work by Nazarov, Treil, Volberg, and then subsequent work by Lacey, Sawyer, Shen, Uriarte-Tuero allow for an answer in terms of the Hilbert transform.

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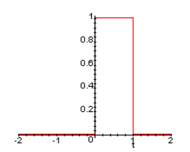

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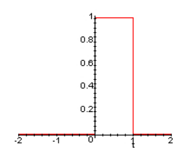

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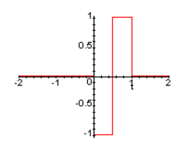

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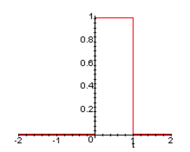

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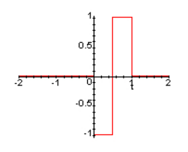

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- $\{h_I\}_{I\in\mathcal{D}}$ is an orthonormal basis of L^2.

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&= \sum_{I\in\mathcal{D}} \langle b, h_I\rangle_{L^2} \langle f, h_I\rangle_{L^2} h_I^1 + \sum_{I\in\mathcal{D}} \langle b, h_I\rangle_{L^2} \left\langle f, h_I^1\right\rangle_{L^2} h_I \\
&\quad + \sum_{I\in\mathcal{D}} \left\langle b, h_I^1\right\rangle_{L^2} \langle f, h_I\rangle_{L^2} h_I.
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Definition (Haar Paraproducts)

Given a symbol sequence $b = \{b_I\}_{I\in\mathcal{D}}$ and a pair $(\alpha,\beta) \in \{0,1\}^2$, define the *dyadic paraproduct* acting on a function f by

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Question (Discrete Sarason Question)

For each choice of pairs $(\alpha, \beta), (\epsilon, \delta) \in \{0,1\}^2$, obtain necessary and sufficient conditions on symbols b and d so that

$$\left\| \mathsf{P}_b^{(\alpha,\beta)} \circ \mathsf{P}_d^{(\epsilon,\delta)} \right\|_{L^2 \to L^2} < \infty.$$

Internal Cancellations and Simple Characterizations

When there are internal zeros the behavior of $\mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)}$reduces to the behavior of $P_a^{(\alpha,\beta)}$ for a special symbol a.

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Here $b \circ d$ is the Schur product of the symbols, i.e., $(b \circ d)_I = b_I d_I$.

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Theorem (Characterization of Type $(1,1)$)

The operator norm $\left\| \mathsf{P}_a^{(1,1)} \right\|_{L^2 \to L^2}$ *of* $\mathsf{P}_a^{(1,1)}$ *on* L^2 *satisfies*

$$\left\| \mathsf{P}_a^{(1,1)} \right\|_{L^2 \to L^2} \approx \left\| \widehat{S}(a) \right\|_{CM} + \|E(a)\|_{\ell^\infty} .$$

Alternate Interpretations: Testing Conditions

It is easy to see for paraproducts of type $(0,0)$ that:

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These observations suggest seeking a characterization for the other compositions in terms of testing conditions on classes of functions.

Characterization of Type $(0, 1, 1, 0)$

For a sequence a, and interval $I \in \mathcal{D}$ let $\mathsf{Q}_I a \equiv \sum_{J \subset I} a_J h_J$.

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$$\left\| \mathsf{Q}_I \mathsf{P}_b^{(0,1)} \mathsf{P}_d^{(1,0)} \left(\mathsf{Q}_I \overline{d} \right) \right\|_{L^2}^2 \quad \leq \quad C_1^2 \left\| \mathsf{Q}_I d \right\|_{L^2}^2 \quad \forall I \in \mathcal{D};$$

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Moreover, the norm of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ *on* L^2 *satisfies*

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} \right\|_{L^2 \to L^2} \approx C_1 + C_2$$

where C_1 *and* C_2 *are the best constants appearing above.*

Rephrasing the Testing Conditions

We want to rephrase the testing conditions on $\mathsf{Q}_I\overline{d}$ and $\mathsf{Q}_I\overline{b}$:

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It isn't hard to see that these are equivalent to the following inequalities on the sequences:

$$\begin{aligned}\sum_{J\subset I}\left|b_J\right|^2\frac{1}{\left|J\right|^2}\left(\sum_{L\subset J}\left|d_L\right|^2\right)^2 &\leq C_1^2\sum_{L\subset I}\left|d_L\right|^2 \quad \forall I\in\mathcal{D};\\ \sum_{J\subset I}\left|d_J\right|^2\frac{1}{\left|J\right|^2}\left(\sum_{L\subset J}\left|b_L\right|^2\right)^2 &\leq C_2^2\sum_{L\subset I}\left|b_L\right|^2 \quad \forall I\in\mathcal{D}.\end{aligned}$$

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Again, it is possible to recast the conditions:

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as expressions depending only on the sequences. In particular, these are equivalent to the following inequalities:

$$\begin{aligned}
\frac{\left|d_I\right|^2}{|I|} \sum_{L \subsetneq I} \left|b_L\right|^2 &\leq C_1^2 \quad \forall I \in \mathcal{D}; \\
\sum_{J \subset I} \frac{\left|d_J\right|^2}{|J|} \left(\sum_{K \subset J_+} \left|b_K\right|^2 - \sum_{K \subset J_-} \left|b_K\right|^2 \right)^2 &\leq C_2^2 \sum_{L \subset I} \left|b_L\right|^2 \quad \forall I \in \mathcal{D}.
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Functions Constant on Tiles

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Easy to show:

$$\left\{ \widetilde{\mathbf{1}}_{T(I)} \right\}_{I \in \mathcal{D}} \text{ is an orthonormal basis of } L_c^2(\mathcal{H});$$

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The Gram Matrix of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$

Let $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}} = [G_{I,J}]_{I,J\in\mathcal{D}}$ be the Gram matrix of the operator $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ relative to the Haar basis $\{h_I\}_{I\in\mathcal{D}}$.

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G_{I,J} &= \left\langle \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} h_J, h_I \right\rangle_{L^2} = \left\langle \mathsf{P}_d^{(1,0)} h_J, \mathsf{P}_b^{(1,0)} h_I \right\rangle_{L^2} \\
&= \left\langle d_J \, h_J^1, b_I \, h_I^1 \right\rangle_{L^2} \\
&= \overline{b_I} d_J \frac{|I \cap J|}{|I|\,|J|} = \begin{cases} \overline{b_I} d_J \frac{1}{|I|} & \text{if} \quad J \subset I \\ \overline{b_I} d_J \frac{1}{|J|} & \text{if} \quad I \subset J \\ 0 & \text{if} \quad I \cap J = \emptyset. \end{cases}
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Idea: Construct $\mathsf{T}_{b,d}^{(0,1,1,0)} : L_c^2(\mathcal{H}) \to L_c^2(\mathcal{H})$ that has the same Gram matrix as $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$, but with respect to the basis $\left\{ \widetilde{\mathbf{1}}_{T(I)} \right\}_{I \in \mathcal{D}}$.

The Operator $\mathsf{T}_{b,d}^{(0,1,1,0)}$ and it Gram Matrix

For $\lambda \in \mathbb{R}$ and $a = \{a_I\}_{I \in \mathcal{D}}$ the multiplication operator $\mathcal{M}_a^\lambda$ is defined on basis elements $\widetilde{\mathbf{1}}_{T(K)}$ by

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Connecting the Problem to a Two Weight Inequality

Up to an absolute constant, $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,1,0)}}$ matches $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}}$ in the lower triangle where $J \subset I$.

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The inequality we wish to characterize is

$$\left\| \mathcal{M}^{0}_{b} \mathsf{U} \mathcal{M}^{-1}_{d} f \right\|_{L^2_c(\mathcal{H})} = \left\| \mathsf{T}^{(0,1,1,0)}_{b,d} f \right\|_{L^2_c(\mathcal{H})} \lesssim \| f \|_{L^2_c(\mathcal{H})} .$$

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Define U on $L_c^2(\mathcal{H})$, where

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For appropriate choice of weights σ and w on $\mathcal{H}$ the desired estimate is simply:

$$\|\mathsf{U}(\sigma k)\|_{L_c^2(\mathcal{H};w)} \lesssim \|k\|_{L_c^2(\mathcal{H};\sigma)}.$$

A Two Weight Theorem for Positive Operators

Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

Let w and σ be non-negative weights on $\mathcal{H}$. Then

$$\mathsf{U}(\sigma\cdot) : L^2(\mathcal{H};\sigma) \to L^2(\mathcal{H};w)$$

is bounded if and only if the following testing condition holds:

$$\left\|\mathbf{1}_{Q(I)}\mathsf{U}\left(\sigma\mathbf{1}_{Q(I)}\right)\right\|^2_{L^2(\mathcal{H};w)} \leq C_0^2 \left\|\mathbf{1}_{Q(I)}\right\|^2_{L^2(\mathcal{H};\sigma)}.$$

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- Appropriate choice of w and σ will then provide the backward testing condition when studying $T_{d,b}^{(0,1,1,0)}$.

The Gram Matrix of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$

Let $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ be the Gram matrix of the operator $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$ relative to the Haar basis $\{h_I\}_{I \in \mathcal{D}}$.

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$$\begin{aligned} \left\| \mathbf{1}_{Q_\pm(K)} \right\|_{L^2(\mathcal{H})} &= \frac{|K|}{2}; \\ \mathcal{M}_a^\lambda \mathbf{1}_{Q_\pm(K)} &= -\sum_{L \subset K_-} a_L \left|L\right|^\lambda \mathbf{1}_{T(L)} + \sum_{L \subset K_+} a_L \left|L\right|^\lambda \mathbf{1}_{T(L)}. \end{aligned}$$

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The Gram matrix $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,0,0)}} = \left[G_{I,J}\right]_{I,J\in\mathcal{D}}$ of $\mathsf{T}_{b,d}^{(0,1,0,0)}$ relative to the basis $\left\{\widetilde{\mathbf{1}}_{T(I)}\right\}_{I\in\mathcal{D}}$ then has entries given by

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The Gram Matrix for the Operator $\mathsf{T}_{b,d}^{(0,1,0,0)}$

The Gram matrix $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,0,0)}} = [G_{I,J}]_{I,J\in\mathcal{D}}$ of $\mathsf{T}_{b,d}^{(0,1,0,0)}$ relative to the basis $\left\{\widetilde{\mathbf{1}}_{T(I)}\right\}_{I\in\mathcal{D}}$ then has entries given by

$$\begin{aligned} G_{I,J} &= \left\langle \mathsf{T}_{b,d}^{(0,1,0,0)}\widetilde{\mathbf{1}}_{T(J)}, \widetilde{\mathbf{1}}_{T(I)} \right\rangle_{L^2(\mathcal{H})} \\ &= \sqrt{2}\begin{cases} -\overline{b_I}\, d_J\, |J|^{-\frac{1}{2}} & \text{if} \quad I \subset J_- \\ \overline{b_I}\, d_J\, |J|^{-\frac{1}{2}} & \text{if} \quad I \subset J_+ \\ 0 & \text{if} \quad J \subset I \text{ or } I\cap J = \emptyset. \end{cases} \end{aligned}$$

Thus, up to an absolute constant, $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,0,0)}} = \mathfrak{G}_{\mathsf{P}_b^{(0,1)}\circ \mathsf{P}_d^{(0,0)}}$, and so

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)} \right\|_{L^2\to L^2} \approx \left\| \mathsf{T}_{b,d}^{(0,1,0,0)} \right\|_{L^2(\mathcal{H})\to L^2(\mathcal{H})} .$$

Connecting to a Two Weight Inequality

The inequality we wish to characterize is:

$$\left\| \mathcal{M}_{\overline{b}}^{-1} \mathsf{U} \mathcal{M}_{d}^{\frac{1}{2}} f \right\|_{L^2_c(\mathcal{H})} = \left\| \mathsf{T}_{b,d}^{(0,1,0,0)} f \right\|_{L^2_c(\mathcal{H})} \lesssim \|f\|_{L^2_c(\mathcal{H})}.$$

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One sees that the inequality to be characterized is equivalent to:

$$\|\mathsf{U}(\mu g)\|_{L^2_c(\mathcal{H};\nu)} \lesssim \|g\|_{L^2_c(\mathcal{H};\mu)},$$

where the weights μ and ν are given by

$$\begin{aligned} \nu &\equiv \sum_{I \in \mathcal{D}} |b_I|^2 |I|^{-2} \mathbf{1}_{T(I)} \\ \mu &\equiv \sum_{I \in \mathcal{D}} |d_I|^{-2} |I|^{-1} \mathbf{1}_{T(I)}. \end{aligned}$$

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$$\mathsf{U}\left(\mu\cdot\right) : L^2_c\left(\mathcal{H};\mu\right) \to L^2_c\left(\mathcal{H};\nu\right)$$

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$$\begin{aligned} \left\| \mathsf{U}\left(\mu \mathbf{1}_{T(I)}\right)\right\|_{L^2_c(\mathcal{H};\nu)} &\leq C_1 \left\|\mathbf{1}_{T(I)}\right\|_{L^2_c(\mathcal{H};\mu)} = \sqrt{\mu\left(T\left(I\right)\right)}, \\ \left\|\mathbf{1}_{Q(I)}\mathsf{U}^*\left(\nu \mathbf{1}_{Q(I)}\right)\right\|_{L^2_c(\mathcal{H};\mu)} &\leq C_2 \left\|\mathbf{1}_{Q(I)}\right\|_{L^2_c(\mathcal{H};\nu)} = \sqrt{\nu\left(Q\left(I\right)\right)}, \end{aligned}$$

hold for all $I \in \mathcal{D}$.

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\end{aligned}$$

hold for all $I \in \mathcal{D}$. Moreover, we have that

$$\|\mathsf{U}\|_{L^2_c(\mathcal{H};\mu)\to L^2_c(\mathcal{H};\nu)} \approx C_1 + C_2$$

where C_1 and C_2 are the best constants appearing above.

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Question

For each $I \in \mathcal{D}$ determine function $F_I, B_I \in L^2$ of norm 1 such that $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,1)}$ is bounded on L^2 if and only if

$$\begin{aligned} \left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,1)} F_I \right\|_{L^2} & \leq & C_1 \quad \forall I \in \mathcal{D}; \\ \left\| \mathsf{P}_d^{(1,0)} \circ \mathsf{P}_b^{(1,0)} B_I \right\|_{L^2} & \leq & C_2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

Moreover, we will have

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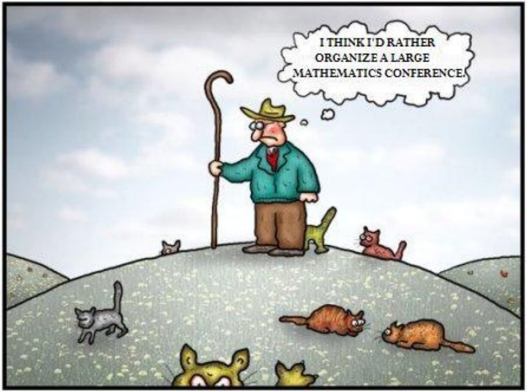
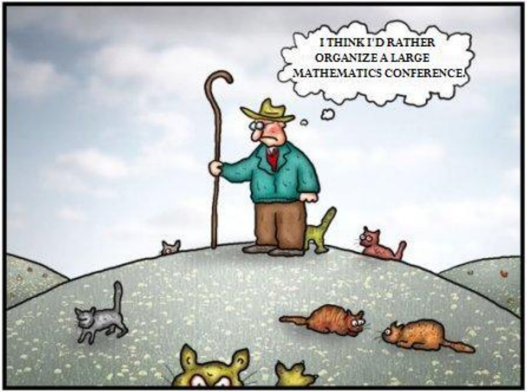


The daydreams of cat herders



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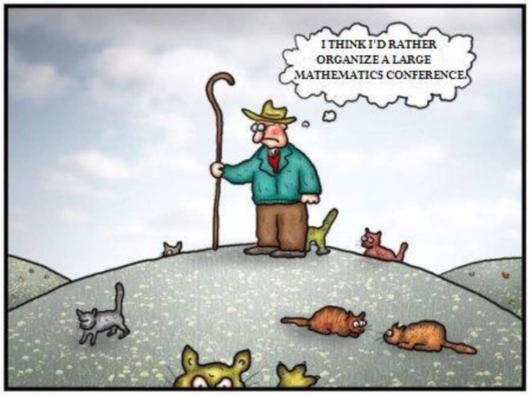

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Thanks to Nicola, John, Marco, Stefan, and Maura for Organizing the Meeting!

Thank You!