Composition of Haar Paraproducts

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Hilbert Function Spaces
Gargnano sul Garda
May 20 – 24, 2013
This talk is based on joint work with:

- Eric T. Sawyer
- McMaster University
- Sandra Pott
- Lund University
- Maria Reguera Rodriguez
- Universidad Autónoma de Barcelona
- B. D. Wick (Georgia Tech)
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Sarason’s Conjecture

- $H^2(\mathbb{D})$, the standard Hardy space on $\mathbb{D}$.
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- $\mathbb{P} : L^2(\mathbb{T}) \rightarrow H^2(\mathbb{D})$ be the orthogonal projection.
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- A Toeplitz operator with symbol $\varphi$ is the following map from $H^2(\mathbb{D}) \to H^2(\mathbb{D})$:
  \[ T_\varphi(f) \equiv P (\varphi f). \]
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- An important question raised by Sarason is the following:

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Conjecture (Sarason Conjecture)
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The composition of $T\varphi T\psi$ is bounded on $H^2(\mathbb{D})$ if and only if

$$\sup_{z \in \mathbb{D}} \left( \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - z\bar{\xi}|^2} |\varphi(\xi)|^2\, dm(\xi) \right) \left( \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - z\bar{\xi}|^2} |\psi(\xi)|^2\, dm(\xi) \right) < \infty$$
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Unfortunately, this is not true!
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Unfortunately, this is not true! A counterexample was constructed by Nazarov.
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**Question (Sarason Question (Revised Version))**

*Obtain necessary and sufficient (testable ?) conditions so that one can tell if $T_\varphi T_\psi$ is bounded on $H^2(\mathbb{D})$ by evaluating these conditions.*
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Possible to rephrase this question as one about the two-weight boundedness of the Hilbert transform. Deep work by Nazarov, Treil, Volberg, and then subsequent work by Lacey, Sawyer, Shen, Uriarte-Tuero allow for an answer in terms of the Hilbert transform.
Haar Paraproducts

- $L^2 \equiv L^2(\mathbb{R})$;
Haar Paraproducts

- \( L^2 \equiv L^2(\mathbb{R}); \)
- \( \mathcal{D} \) is the standard grid of dyadic intervals on \( \mathbb{R}; \)
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- $L^2 \equiv L^2(\mathbb{R})$;
- $\mathcal{D}$ is the standard grid of dyadic intervals on $\mathbb{R}$;
- Define the Haar function $h^0_I$ and averaging function $h^1_I$ by
  \[
  h^0_I \equiv h_I \equiv \frac{1}{\sqrt{|I|}} (-1_{I_-} + 1_{I_+}) \quad I \in \mathcal{D}
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Haar Paraproducts from Multiplication Operators

Given a function $b$ and $f$ it is possible to study their pointwise product by expanding in their Haar series:
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$$bf = \left( \sum_{I \in \mathcal{D}} \langle b, h_I \rangle_{L^2} h_I \right) \left( \sum_{J \in \mathcal{D}} \langle f, h_J \rangle_{L^2} h_J \right)$$
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$$= \left( \sum_{I = J} + \sum_{I \subsetneq J} + \sum_{J \subsetneq I} \right) \langle b, h_I \rangle_{L^2} \langle f, h_J \rangle_{L^2} h_I h_J$$
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$$= \sum_{I \in \mathcal{D}} \langle b, h_I \rangle_{L^2} \langle f, h_I \rangle_{L^2} h_I^1 + \sum_{I \in \mathcal{D}} \langle b, h_I \rangle_{L^2} \langle f, h_I^1 \rangle_{L^2} h_I$$

$$+ \sum_{I \in \mathcal{D}} \langle b, h_I^1 \rangle_{L^2} \langle f, h_I \rangle_{L^2} h_I.$$
Haar Paraproducts

Definition (Haar Paraproducts)

Given a symbol sequence \( b = \{b_I\}_{I \in \mathcal{D}} \) and a pair \((\alpha, \beta) \in \{0, 1\}^2\), define the \textit{dyadic paraproduct} acting on a function \( f \) by

\[
P_b^{(\alpha, \beta)} f \equiv \sum_{I \in \mathcal{D}} b_I \left\langle f, h_I^\beta \right\rangle_{L^2} h_I^\alpha.
\]
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$$P_b^{(\alpha, \beta)} f \equiv \sum_{I \in \mathcal{D}} b_I \langle f, h^\beta_I \rangle_{L^2} h^\alpha_I.$$ 

The index $(\alpha, \beta)$ is referred to as the \textit{type} of $P_b^{(\alpha, \beta)}$. 
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\]

The index \((\alpha, \beta)\) is referred to as the type of \( P_b^{(\alpha, \beta)} \).

Question (Discrete Sarason Question)

*For each choice of pairs \((\alpha, \beta), (\epsilon, \delta) \in \{0, 1\}^2\), obtain necessary and sufficient conditions on symbols \( b \) and \( d \) so that*

\[
\left\| P_b^{(\alpha, \beta)} \circ P_d^{(\epsilon, \delta)} \right\|_{L^2 \to L^2} < \infty.
\]
Internal Cancellations and Simple Characterizations

When there are internal zeros the behavior of $P_b^{(\alpha,0)} \circ P_d^{(0,\beta)}$ reduces to the behavior of $P_a^{(\alpha,\beta)}$ for a special symbol $a$. 
Internal Cancellations and Simple Characterizations

When there are internal zeros the behavior of $P_b^{(\alpha,0)} \circ P_d^{(0,\beta)}$ reduces to the behavior of $P_a^{(\alpha,\beta)}$ for a special symbol $a$. For $f, g \in L^2$, let $f \otimes g : L^2 \to L^2$ be the map given by

$$f \otimes g(h) \equiv f \langle g, h \rangle_{L^2}.$$
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Then:

$$P_b^{(\alpha,0)} \circ P_d^{(0,\beta)} = \left( \sum_{I \in \mathcal{D}} b_I h_I^\alpha \otimes h_I \right) \left( \sum_{J \in \mathcal{D}} d_J h_J \otimes h_J^\beta \right)$$
Motivations  Classical Characterizations

Internal Cancellations and Simple Characterizations

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Then:

$$P^{(\alpha,0)}_b \circ P^{(0,\beta)}_d = \left( \sum_{I \in \mathcal{D}} b_I h^\alpha_I \otimes h_I \right) \left( \sum_{J \in \mathcal{D}} d_J h_J \otimes h^\beta_J \right)$$

$$= \sum_{I \in \mathcal{D}} b_I d_I h^\alpha_I \otimes h^\beta_I$$

$$= P^{(\alpha,\beta)}_{b \circ d}.$$
Internal Cancellations and Simple Characterizations

When there are internal zeros the behavior of $P_{b}^{(\alpha,0)} \circ P_{d}^{(0,\beta)}$ reduces to the behavior of $P_{a}^{(\alpha,\beta)}$ for a special symbol $a$. For $f, g \in L^2$, let $f \otimes g : L^2 \to L^2$ be the map given by

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Then:

$$P_{b}^{(\alpha,0)} \circ P_{d}^{(0,\beta)} = \left( \sum_{I \in \mathcal{D}} b_I h_I^{\alpha} \otimes h_I \right) \left( \sum_{J \in \mathcal{D}} d_J h_J^{\beta} \otimes h_J^{\beta} \right)$$

$$= \sum_{I \in \mathcal{D}} b_I d_I h_I^{\alpha} \otimes h_I^{\beta}$$

$$= P_{b \circ d}^{(\alpha,\beta)}.$$ 

Here $b \circ d$ is the Schur product of the symbols, i.e., $(b \circ d)_I = b_I d_I$. 

B. D. Wick (Georgia Tech)  Composition of Haar Paraproducts  HFS  7 / 29
Norms and Induced Sequences

For a sequence $a = \{a_I\}_{I \in \mathcal{D}}$ define the following quantities:

$$
\|a\|_{\ell_\infty} \equiv \sup_{I \in \mathcal{D}} |a_I|;
$$

$$
\|a\|_{CM} \equiv \sup_{I \in \mathcal{D}} \sum_{J \subset I} |a_J|^2.
$$

Associate to $\{a_I\}_{I \in \mathcal{D}}$ two additional sequences indexed by $\mathcal{D}$:

$$
E(a) \equiv \begin{cases} 1 & |I| \sum_{J \subset I} a_J \\ I \in \mathcal{D} \end{cases};
$$

$$
S(a) \equiv \begin{cases} \langle \sum_{J \in \mathcal{D}} a_J h^*_J, h_I \rangle_{L_2} \\ I \in \mathcal{D} \end{cases} = \begin{cases} \sum_{J \varsubsetneq I} a_J h^*_J(I) \\ I \in \mathcal{D} \end{cases}.
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$$

Associate to $\{a_I\}_{I \in \mathcal{D}}$ two additional sequences indexed by $\mathcal{D}$:

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E(a) \equiv \left\{ \frac{1}{|I|} \sum_{J \subset I} a_J \right\}_{I \in \mathcal{D}} ;
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$$E(a) \equiv \left\{ \frac{1}{|I|} \sum_{J \subset I} a_J \right\}_{I \in \mathcal{D}};$$

$$\widehat{S}(a) \equiv \left\{ \left\langle \sum_{J \in \mathcal{D}} a_J h_J^1, h_I \right\rangle_{L^2} \right\}_{I \in \mathcal{D}} = \left\{ \sum_{J \subsetneq I} a_J \widehat{h}_J^1 (I) \right\}_{I \in \mathcal{D}}.$$
Classical Characterizations

Theorem (Characterizations of Type $(0,0)$, $(0,1)$, and $(1,0)$)

The following characterizations are true:

The operator norm $\|P_{(1,1)}a\|_{L^2 \to L^2}$ of $P_{(1,1)}a$ on $L^2$ satisfies

$$\|P_{(1,1)}a\|_{L^2 \to L^2} \approx \|S(a)\|_{CM} + \|E(a)\|_{\ell^\infty}.$$
Classical Characterizations

Theorem (Characterizations of Type $(0, 0)$, $(0, 1)$, and $(1, 0)$)

The following characterizations are true:

\[ \| P^{(0,0)}_a \|_{L^2 \to L^2} = \| a \|_{\ell^\infty}; \]
Classical Characterizations

Theorem (Characterizations of Type \((0,0), (0, 1),\) and \((1,0))

The following characterizations are true:

\[
\left\| P_{a}^{(0,0)} \right\|_{L^2 \rightarrow L^2} = \| a \|_{\ell^\infty} ;
\]
\[
\left\| P_{a}^{(0,1)} \right\|_{L^2 \rightarrow L^2} = \left\| P_{a}^{(1,0)} \right\|_{L^2 \rightarrow L^2} \approx \| a \|_{CM} .
\]
Theorem (Characterizations of Type \((0, 0), (0, 1), \text{ and } (1, 0)\))

The following characterizations are true:

\[ \left\| P^{(0,0)}_a \right\|_{L^2 \to L^2} = \| a \|_{\ell^\infty} ; \]
\[ \left\| P^{(0,1)}_a \right\|_{L^2 \to L^2} = \left\| P^{(1,0)}_a \right\|_{L^2 \to L^2} \approx \| a \|_{CM} . \]

\[ P^{(1,1)}_a = P^{(1,0)}_{\hat{S}(a)} + P^{(0,1)}_{\hat{S}(a)} + P^{(0,0)}_{E(a)} . \]
Classical Characterizations

**Theorem (Characterizations of Type \((0, 0), (0, 1), \text{and} (1, 0)\))**

The following characterizations are true:

\[
\| P_a^{(0,0)} \|_{L^2 \to L^2} = \| a \|_{\ell^\infty} ;
\]

\[
\| P_a^{(0,1)} \|_{L^2 \to L^2} = \| P_a^{(1,0)} \|_{L^2 \to L^2} \approx \| a \|_{CM} .
\]

\[
P_a^{(1,1)} = \hat{P}_{S(a)}^{(1,0)} + \hat{P}_{S(a)}^{(0,1)} + \hat{P}_{E(a)}^{(0,0)} .
\]

**Theorem (Characterization of Type \((1, 1)\))**

The operator norm \(\| P_a^{(1,1)} \|_{L^2 \to L^2}\) of \(P_a^{(1,1)}\) on \(L^2\) satisfies

\[
\| P_a^{(1,1)} \|_{L^2 \to L^2} \approx \| \hat{S}(a) \|_{CM} + \| E(a) \|_{\ell^\infty} .
\]
Alternate Interpretations: Testing Conditions

It is easy to see for paraproducts of type \((0, 0)\) that:

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\left\| P^{(0,0)}_a \right\|_{L^2 ightarrow L^2} = \| a \|_{\ell^\infty}
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Alternate Interpretations: Testing Conditions

It is easy to see for paraproducts of type \((0, 0)\) that:

\[
\left\| P_a^{(0,0)} \right\|_{L^2 \to L^2} = \left\| a \right\|_{\ell^\infty} = \sup_{I \in \mathcal{D}} \left\| P_a^{(0,0)} h_I \right\|_{L^2}.
\]
Alternate Interpretations: Testing Conditions

It is easy to see for paraproducts of type $(0,0)$ that:

\[ \left\| P^{(0,0)}_{a} \right\|_{L^2 	o L^2} = \| a \|_{\ell^\infty} = \sup_{I \in \mathcal{D}} \left\| P^{(0,0)}_{a} h_I \right\|_{L^2}. \]

Moreover,

\[ \left\| P^{(1,0)}_{a} \right\|_{L^2 	o L^2} \approx \left\| P^{(0,1)}_{a} \right\|_{L^2 	o L^2} \approx \| a \|_{CM}. \]
Alternate Interpretations: Testing Conditions

It is easy to see for paraproducts of type \((0,0)\) that:

\[
\left\| P^{(0,0)}_a \right\|_{L^2 \to L^2} = a_{\ell\infty} = \sup_{I \in \mathcal{D}} \left\| P^{(0,0)}_a h_I \right\|_{L^2}.
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It is easy to see for paraproducts of type $(0,0)$ that:

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$$

Moreover,

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\left\| P_a^{(1,0)} \right\|_{L^2 \to L^2} \approx \left\| P_a^{(0,1)} \right\|_{L^2 \to L^2} \approx \| a \|_{CM} \approx \sup_{I \in D} \left\| P_a^{(0,1)} h_I \right\|_{L^2}.
$$

These observations suggest seeking a characterization for the other compositions in terms of testing conditions on classes of functions.
Characterization of Type \((0, 1, 1, 0)\)

For a sequence \(a\), and interval \(I \in \mathcal{D}\) let \(Q_I a \equiv \sum_{J \subset I} a_J h_J\).
Characterization of Type \((0, 1, 1, 0)\)

For a sequence \(a\), and interval \(I \in \mathcal{D}\) let \(Q_I a \equiv \sum_{J \subset I} a_J h_J\).

**Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)**

\[
\|Q_I a\|_{L^2} \leq C_1 \|Q_I b\|_{L^2} \quad \text{for all } I \in \mathcal{D},
\]

where \(C_1\) and \(C_2\) are the best constants appearing above.
Characterization of Type \((0, 1, 1, 0)\)

For a sequence \(a\), and interval \(I \in \mathcal{D}\) let \(Q_I a \equiv \sum_{J \subset I} a_J h_J\).

**Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)**

The composition \(P_b^{(0, 1)} \circ P_d^{(1, 0)}\) is bounded on \(L^2\) if and only if both
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The composition \(P_b^{(0,1)} \circ P_d^{(1,0)}\) is bounded on \(L^2\) if and only if both

\[
\left\|Q_I P_b^{(0,1)} P_d^{(1,0)} (Q_I \overline{d})\right\|_{L^2}^2 \leq C_1^2 \left\|Q_I d\right\|_{L^2}^2 \quad \forall I \in \mathcal{D};
\]
Characterization of Type \((0, 1, 1, 0)\)

For a sequence \(a\), and interval \(I \in \mathcal{D}\) let \(Q_I a \equiv \sum_{J \subseteq I} a_J h_J\).

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\]

\[
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\[
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\]
\[
\left\| Q_I P_d^{(0,1)} P_b^{(1,0)} (Q_I \overline{b}) \right\|_{L^2}^2 \leq C_2^2 \left\| Q_I b \right\|_{L^2}^2 \forall I \in \mathcal{D}.
\]

Moreover, the norm of $P_b^{(0,1)} \circ P_d^{(1,0)}$ on $L^2$ satisfies

\[
\left\| P_b^{(0,1)} \circ P_d^{(1,0)} \right\|_{L^2 \rightarrow L^2} \approx C_1 + C_2
\]

where $C_1$ and $C_2$ are the best constants appearing above.
Rephrasing the Testing Conditions

We want to rephrase the testing conditions on $Q_I \overline{d}$ and $Q_I \overline{b}$:

$$
\left\| Q_I P^{(0,1)}_b P^{(1,0)}_d (Q_I \overline{d}) \right\|_{L^2}^2 \leq C_1^2 \left\| Q_I d \right\|_{L^2}^2 \quad \forall I \in D;
$$

$$
\left\| Q_I P^{(0,1)}_d P^{(1,0)}_b (Q_I \overline{b}) \right\|_{L^2}^2 \leq C_2^2 \left\| Q_I b \right\|_{L^2}^2 \quad \forall I \in D.
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\]
\[
\left\| Q_I P_d^{(0,1)} P_b^{(1,0)} (Q_I \overline{b}) \right\|_{L^2}^2 \leq C_2^2 \left\| Q_I b \right\|_{L^2}^2 \quad \forall I \in D.
\]

It isn’t hard to see that these are equivalent to the following inequalities on the sequences:

\[
\sum_{J \subset I} |b_J|^2 \frac{1}{|J|^2} \left( \sum_{L \subset J} |d_L|^2 \right)^2 \leq C_1^2 \sum_{L \subset I} |d_L|^2 \quad \forall I \in D;
\]
\[
\sum_{J \subset I} |d_J|^2 \frac{1}{|J|^2} \left( \sum_{L \subset J} |b_L|^2 \right)^2 \leq C_2^2 \sum_{L \subset I} |b_L|^2 \quad \forall I \in D.
\]
Characterization of Type (0, 1, 0, 0)

Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)

\[
\text{The composition } P(0, 1) \circ P(0, 0) \text{ is bounded on } L^2 \text{ if and only if both } \left\| d \right\|_{L^2} \leq C_1 \forall I \in D; \left\| Q I P(0, 0) d P(1, 0) Q I \right\|_{L^2} \leq C_2 \left\| Q I \right\|_{L^2} \forall I \in D. \]

Moreover, the norm of \( P(0, 1) \circ P(0, 0) \) on \( L^2 \) satisfies

\[
\left\| P(0, 1) \circ P(0, 0) \right\|_{L^2 \to L^2} \approx C_1 + C_2
\]

where \( C_1 \) and \( C_2 \) are the best constants appearing above.
Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)

The composition $P_b^{(0,1)} \circ P_d^{(0,0)}$ is bounded on $L^2$ if and only if both

$$\|P_b^{(0,1)} \circ P_d^{(0,0)}\|_{L^2 \to L^2} \leq C_1 + C_2 \quad \forall I \in D,$$

where $C_1$ and $C_2$ are the best constants appearing above.
Characterization of Type \((0, 1, 0, 0)\)

**Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)**

The composition \(P_{b}^{(0,1)} \circ P_{d}^{(0,0)}\) is bounded on \(L^2\) if and only if both

\[|d_I|^2 \left\| P_{b}^{(0,1)} h_I \right\|^2_{L^2} \leq C_2^2 \forall I \in \mathcal{D};\]

where \(C_1^2\) and \(C_2^2\) are the best constants appearing above.
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\[
|d_I|^2 \left\| P_b^{(0,1)} h_I \right\|^2_{L^2} \leq C_1^2 \quad \forall I \in D;
\]

\[
\left\| Q_I P_d^{(0,0)} P_b^{(1,0)} Q_I \bar{b} \right\|^2_{L^2} \leq C_2^2 \left\| Q_I b \right\|^2_{L^2} \quad \forall I \in D.
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Characterization of Type \((0, 1, 0, 0)\)

Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)

The composition \(P_{(0,1)} \circ P_{(0,0)}\) is bounded on \(L^2\) if and only if both

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|d_I|^2 \left\| P_{(0,1)} h_I \right\|^2_{L^2} \leq C_1^2 \quad \forall I \in D;
\]

\[
\left\| Q_I P_{(0,0)} P_{(1,0)} Q_I b \right\|^2_{L^2} \leq C_2^2 \left\| Q_I b \right\|^2_{L^2} \quad \forall I \in D.
\]

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Rephrasing Testing Conditions

Again, it is possible to recast the conditions:

\[
|d_I|^2 \left\| P_b^{(0,1)} h_I \right\|_{L^2}^2 \leq C_1^2 \quad \forall I \in \mathcal{D};
\]

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as expressions depending only on the sequences.
Rephrasing Testing Conditions

Again, it is possible to recast the conditions:

\[
|d_I|^2 \left\| P^{(0,1)}_b h_I \right\|_{L^2}^2 \leq C_1^2 \quad \forall I \in \mathcal{D};
\]
\[
\left\| QIP^{(0,0)}_d P^{(1,0)}_b Q_I \right\|_{L^2}^2 \leq C_2^2 \left\| Q_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}
\]

as expressions depending only on the sequences. In particular, these are equivalent to the following inequalities:

\[
\frac{|d_I|^2}{|I|} \sum_{L \subsetneq I} |b_L|^2 \leq C_1^2 \quad \forall I \in \mathcal{D};
\]
\[
\sum_{J \subset I} \frac{|d_J|^2}{|J|} \left( \sum_{K \subset J_+} |b_K|^2 - \sum_{K \subset J_-} |b_K|^2 \right)^2 \leq C_2^2 \sum_{L \subset I} |b_L|^2 \quad \forall I \in \mathcal{D}.
\]
Preliminaries

For $I \in \mathcal{D}$ set

- The dyadic lattice $\mathcal{D}$ is in correspondence with the Carleson Tiles.
- Let $\mathcal{H}$ denote the upper half plane $\mathbb{C}^+$:
  \[ \mathcal{H} = \bigcup_{I \in \mathcal{D}} T(I) \]
- For a non-negative function $\sigma$ let $L^2(\mathcal{H}; \sigma)$ denote the functions that are square integrable with respect to $\sigma dA$, i.e,
  \[ \|f\|_{L^2(\mathcal{H}; \sigma)} \equiv \int_{\mathcal{H}} |f(z)|^2 \sigma(z) dA(z) < \infty. \]
  When $\sigma \equiv 1$, $L^2(\mathcal{H}; 1) \equiv L^2(\mathcal{H})$.
- For $f \in L^2(\mathcal{H})$, let $\mathcal{H}f \equiv \frac{f}{\|f\|_{L^2(\mathcal{H})}}$ denote the normalized function.
For $I \in \mathcal{D}$ set

$$T(I) \equiv I \times \left[ \frac{|I|}{2}, |I| \right] \quad \text{(Carleson Tile)};$$
Proofs of Main Results

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$$\|f\|_{L^2(\mathcal{H}; \sigma)}^2 \equiv \int_{\mathcal{H}} |f(z)|^2 \sigma(z) \, dA(z) < \infty.$$
Proofs of Main Results

Preliminaries

For $I \in D$ set

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\[\text{B. D. Wick (Georgia Tech)}\]
Preliminaries

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Functions Constant on Tiles

Let $L^2_c(\mathcal{H}) \subset L^2(\mathcal{H})$ be the subspace of functions which are constant on tiles.
Functions Constant on Tiles

Let $L^2_c(\mathcal{H}) \subset L^2(\mathcal{H})$ be the subspace of functions which are constant on tiles. Namely, $f : \mathcal{D} \to \mathbb{C}$

$$f = \sum_{I \in \mathcal{D}} f_I 1_{T(I)}.$$

Easy to show:

* \{ 1_{T(I)} \} \quad I \in \mathcal{D} is an orthonormal basis of $L^2_c(\mathcal{H})$;
* \{ 1_{Q(I)} \} \quad I \in \mathcal{D} is an Riesz basis of $L^2_c(\mathcal{H})$. 
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Then

$$L^2_c(\mathcal{H}) \equiv \left\{ f : \mathcal{D} \to \mathbb{C} : \sum_{I \in \mathcal{D}} |f(I)|^2 |I|^2 < \infty \right\};$$
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$$\|f\|_{L^2_c(\mathcal{H})}^2 \equiv \frac{1}{2} \sum_{I \in \mathcal{D}} |f(I)|^2 |I|^2.$$
Functions Constant on Tiles

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is an Riesz basis of $L^2_c(\mathcal{H})$. 
The Gram Matrix of $P_{b}^{(0,1)} \circ P_{d}^{(1,0)}$

Let $S_{P_{b}^{(0,1)} \circ P_{d}^{(1,0)}} = [G_{I,J}]_{I,J \in D}$ be the Gram matrix of the operator $P_{b}^{(0,1)} \circ P_{d}^{(1,0)}$ relative to the Haar basis $\{h_{I}\}_{I \in D}$. 
The Gram Matrix of $P_b^{(0,1)} \circ P_d^{(1,0)}$

Let $\mathcal{G}_{P_b^{(0,1)} \circ P_d^{(1,0)}} = [G_{I,J}]_{I,J \in D}$ be the Gram matrix of the operator $P_b^{(0,1)} \circ P_d^{(1,0)}$ relative to the Haar basis $\{h_I\}_{I \in D}$. A simple computation shows that it has entries:

$$G_{I,J} = \left\langle P_b^{(0,1)} \circ P_d^{(1,0)} h_J, h_I \right\rangle_{L^2}$$
Let $G_{(0,1)\circ(1,0)} = [G_{I,J}]_{I,J\in D}$ be the Gram matrix of the operator $P_{b}^{(0,1)} \circ P_{d}^{(1,0)}$ relative to the Haar basis $\{h_{I}\}_{I\in D}$. A simple computation show that it has entries:

$$G_{I,J} = \langle P_{b}^{(0,1)} \circ P_{d}^{(1,0)} h_{J}, h_{I} \rangle_{L^2} = \langle P_{d}^{(1,0)} h_{J}, P_{b}^{(1,0)} h_{I} \rangle_{L^2}$$
The Gram Matrix of $P_{b}^{(0,1)} \circ P_{d}^{(1,0)}$

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The Gram Matrix of $P_{b}^{(0,1)} \circ P_{d}^{(1,0)}$

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$$= \langle d_{J} h_{J}^{1}, b_{I} h_{I}^{1} \rangle_{L^{2}}$$

$$= \overline{b_{I}} d_{J} \frac{|I \cap J|}{|I||J|}$$
The Gram Matrix of $P_b^{(0,1)} \circ P_d^{(1,0)}$

Let $G_{P_b^{(0,1)} \circ P_d^{(1,0)}} = [G_{I,J}]_{I,J \in D}$ be the Gram matrix of the operator $P_b^{(0,1)} \circ P_d^{(1,0)}$ relative to the Haar basis $\{h_I\}_{I \in D}$. A simple computation show that it has entries:

\[
G_{I,J} = \langle P_b^{(0,1)} \circ P_d^{(1,0)} h_J, h_I \rangle_{L^2} = \langle P_d^{(1,0)} h_J, P_b^{(1,0)} h_I \rangle_{L^2} = \langle d_J h_J^1, b_I h_I^1 \rangle_{L^2}
\]

\[
= \overline{b_I} d_J \frac{|I \cap J|}{|I||J|} = \begin{cases} 
\overline{b_I} d_J \frac{1}{|I|} & \text{if } J \subset I \\
\overline{b_I} d_J \frac{1}{|J|} & \text{if } I \subset J \\
0 & \text{if } I \cap J = \emptyset.
\end{cases}
\]
The Gram Matrix of $P^{(0,1)}_b \circ P^{(1,0)}_d$

Let $\mathcal{G}_{P^{(0,1)}_b \circ P^{(1,0)}_d} = [G_{I,J}]_{I,J \in D}$ be the Gram matrix of the operator $P^{(0,1)}_b \circ P^{(1,0)}_d$ relative to the Haar basis $\{h_I\}_{I \in D}$. A simple computation show that it has entries:

$$G_{I,J} = \langle P^{(0,1)}_b \circ P^{(1,0)}_d h_J, h_I \rangle_{L^2} = \langle P^{(1,0)}_d h_J, P^{(1,0)}_b h_I \rangle_{L^2} = \langle d_J h^1_J, b_I h^1_I \rangle_{L^2}$$

$$= \overline{b_I} d_J \frac{|I \cap J|}{|I| |J|} = \begin{cases} \overline{b_I} d_J \frac{1}{|I|} & \text{if } J \subset I \\ \overline{b_I} d_J \frac{1}{|J|} & \text{if } I \subset J \\ 0 & \text{if } I \cap J = \emptyset. \end{cases}$$

**Idea:** Construct $T^{(0,1,1,0)}_{b,d} : L^2_c(\mathcal{H}) \to L^2_c(\mathcal{H})$ that has the same Gram matrix as $P^{(0,1)}_b \circ P^{(1,0)}_d$, but with respect to the basis $\{\tilde{1}_{T(I)}\}_{I \in D}$. 
The Operator $T_{b,d}^{(0,1,1,0)}$ and it Gram Matrix

For $\lambda \in \mathbb{R}$ and $a = \{a_I\}_{I \in \mathcal{D}}$ the multiplication operator $M_\lambda^a$ is defined on basis elements $\tilde{1}_{T(K)}$ by

$$M_\lambda^a \tilde{1}_{T(K)} \equiv a_K |K|^\lambda \tilde{1}_{T(K)}.$$
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Define an operator $T_{b,d}^{(0,1,1,0)}$ on $L_c^2(\mathcal{H})$ by

$$T_{b,d}^{(0,1,1,0)} \equiv \mathcal{M}_b^0 \left( \sum_{K \in \mathcal{D}} \tilde{1}_{T(K)} \otimes \tilde{1}_{Q(K)} \right) \mathcal{M}_d^{-1}.$$
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Then the Gram matrix $G_{T_{b,d}^{(0,1,1,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ of $T_{b,d}^{(0,1,1,0)}$ relative to the basis $\{\tilde{1}_{T(I)}\}_{I \in \mathcal{D}}$ has entries

$$G_{I,J} = \left\langle T_{b,d}^{(0,1,1,0)} \tilde{1}_{T(J)}, \tilde{1}_{T(I)} \right\rangle_{L^2(\mathcal{H})}$$.
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Define an operator $\mathbf{T}_{b,d}^{(0,1,1,0)}$ on $L^2_c(\mathcal{H})$ by

$$\mathbf{T}_{b,d}^{(0,1,1,0)} \equiv \mathcal{M}_b^0 \left( \sum_{K \in \mathcal{D}} \tilde{1}_{T(K)} \otimes \tilde{1}_{Q(K)} \right) \mathcal{M}_d^{-1}.$$ 

Then the Gram matrix $G_{\mathbf{T}_{b,d}^{(0,1,1,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ of $\mathbf{T}_{b,d}^{(0,1,1,0)}$ relative to the basis $\{\tilde{1}_{T(I)}\}_{I \in \mathcal{D}}$ has entries

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The Operator $T_{b,d}^{(0,1,1,0)}$ and it Gram Matrix

For $\lambda \in \mathbb{R}$ and $a = \{a_I\}_{I \in \mathcal{D}}$ the multiplication operator $\mathcal{M}_a^\lambda$ is defined on basis elements $\mathcal{I}_T(K)$ by

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$$T_{b,d}^{(0,1,1,0)} \equiv \mathcal{M}_0^b \left( \sum_{K \in \mathcal{D}} \mathcal{I}_T(K) \otimes \mathcal{I}_Q(K) \right) \mathcal{M}_d^{-1}.$$

Then the Gram matrix $\mathcal{G}_T^{(0,1,1,0)} = [G_{I,J}]_{I,J \in \mathcal{D}}$ of $T_{b,d}^{(0,1,1,0)}$ relative to the basis $\{\mathcal{I}_T(I)\}_{I \in \mathcal{D}}$ has entries

$$G_{I,J} = \left\langle T_{b,d}^{(0,1,1,0)} \mathcal{I}_T(J), \mathcal{I}_T(I) \right\rangle_{L^2(\mathcal{H})} = \frac{b_I d_J}{\sqrt{2} |I| |J|^2} \frac{1}{|I|} \frac{1}{|J|} \begin{cases} \frac{1}{|I|} & \text{if } J \subset I \\ 0 & \text{if } J \nsubseteq I \end{cases}.$$
Connecting the Problem to a Two Weight Inequality

Up to an absolute constant, $\mathcal{G}_{T}^{(0,1,1,0)}$ matches $\mathcal{G}_{P}^{(0,1)} \circ P^{(1,0)}$ in the lower triangle where $J \subset I$. 
Connecting the Problem to a Two Weight Inequality

Up to an absolute constant, $\mathcal{G}_{T_{b,d}^{(0,1,1,0)}}$ matches $\mathcal{G}_{P_{b}^{(0,1)} \circ P_{d}^{(1,0)}}$ in the lower triangle where $J \subset I$. So,

$$
\| P_{b}^{(0,1)} \circ P_{d}^{(1,0)} \|_{L^{2} \to L^{2}} \approx \| T_{b,d}^{(0,1,1,0)} \|_{L^{2}(\mathcal{H}) \to L^{2}(\mathcal{H})} + \| T_{d,b}^{(0,1,1,0)} \|_{L^{2}(\mathcal{H}) \to L^{2}(\mathcal{H})} .
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The inequality we wish to characterize is

$$\left\| \mathcal{M}_{b}^{0} \mathcal{M}_{d}^{-1} f \right\|_{L^2_{c}(\mathcal{H})} = \left\| T_{b,d}^{(0,1,1,0)} f \right\|_{L^2_{c}(\mathcal{H})} \lesssim \| f \|_{L^2_{c}(\mathcal{H})}.$$
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\left\| P_{b}^{(0,1)} \circ P_{d}^{(1,0)} \right\|_{L^2 \to L^2} \approx \left\| T_{b,d}^{(0,1,1,0)} \right\|_{L^2(H) \to L^2(H)} + \left\| T_{d,b}^{(0,1,1,0)} \right\|_{L^2(H) \to L^2(H)}.
$$

The inequality we wish to characterize is

$$
\left\| M_{b}^{0} \mathcal{U} M_{d}^{-1} f \right\|_{L_{c}^{2}(H)} = \left\| T_{b,d}^{(0,1,1,0)} f \right\|_{L_{c}^{2}(H)} \lesssim \left\| f \right\|_{L_{c}^{2}(H)}.
$$

Define $\mathcal{U}$ on $L_{c}^{2}(H)$, where

$$
\mathcal{U} \equiv \sum_{K \in \mathcal{D}} \mathbf{1}_{T(K)} \otimes \mathbf{1}_{Q(K)}.
$$
Connecting the Problem to a Two Weight Inequality

Up to an absolute constant, $\mathcal{G}_{T^{(0,1,1,0)}_{b,d}}$ matches $\mathcal{G}_{P^{(0,1)}_{b} \circ P^{(1,0)}_{d}}$ in the lower triangle where $J \subset I$. So,

$$
\left\| P^{(0,1)}_{b} \circ P^{(1,0)}_{d} \right\|_{L^2 \to L^2} \approx \left\| T^{(0,1,1,0)}_{b,d} \right\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})} + \left\| T^{(0,1,1,0)}_{d,b} \right\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})}.
$$

The inequality we wish to characterize is

$$
\left\| \mathcal{M}^{0}_{b} U \mathcal{M}^{-1}_{d} f \right\|_{L^2(\mathcal{H})} = \left\| T^{(0,1,1,0)}_{b,d} f \right\|_{L^2(\mathcal{H})} \lesssim \left\| f \right\|_{L^2(\mathcal{H})}.
$$

Define $U$ on $L^2(\mathcal{H})$, where

$$
U \equiv \sum_{K \in \mathcal{D}} \tilde{1}_{T(K)} \otimes \tilde{1}_{Q(K)}.
$$

For appropriate choice of weights $\sigma$ and $w$ on $\mathcal{H}$ the desired estimate is simply:

$$
\left\| U (\sigma k) \right\|_{L^2(\mathcal{H};w)} \lesssim \left\| k \right\|_{L^2(\mathcal{H};\sigma)}.
$$


A Two Weight Theorem for Positive Operators

Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

Let $w$ and $\sigma$ be non-negative weights on $\mathcal{H}$. Then

$$U(\sigma \cdot) : L^2(\mathcal{H}; \sigma) \rightarrow L^2(\mathcal{H}; w)$$

is bounded if and only if the following testing condition holds:

$$\left\| 1_{Q(I)} U \left( \sigma 1_{Q(I)} \right) \right\|_{L^2(\mathcal{H}; w)}^2 \leq C_0^2 \left\| 1_{Q(I)} \right\|_{L^2(\mathcal{H}; \sigma)}^2.$$
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- The proof of this Theorem is a translation of Sawyer’s proof strategy for two weight inequalities for positive operators.
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- The proof of this Theorem is a translation of Sawyer’s proof strategy for two weight inequalities for positive operators.
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A Two Weight Theorem for Positive Operators

Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

Let $w$ and $\sigma$ be non-negative weights on $\mathcal{H}$. Then

$$
\mathbb{U}(\sigma \cdot) : L^2(\mathcal{H}; \sigma) \rightarrow L^2(\mathcal{H}; w)
$$

is bounded if and only if the following testing condition holds:

$$
\left\| \mathbf{1}_{Q(I)} \mathbb{U} \left( \sigma \mathbf{1}_{Q(I)} \right) \right\|_{L^2(\mathcal{H}; w)}^2 \leq C_0^2 \left\| \mathbf{1}_{Q(I)} \right\|_{L^2(\mathcal{H}; \sigma)}^2.
$$

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- Appropriate choice of $w$ and $\sigma$ will then provide the backward testing condition when studying $T_{d,b}^{(0,1,1,0)}$. 
The Gram Matrix of $P^{(0,1)}_b \circ P^{(0,0)}_d$

Let $\mathcal{G}_{P^{(0,1)}_b \circ P^{(0,0)}_d} = [G_{I,J}]_{I,J \in \mathcal{D}}$ be the Gram matrix of the operator $P^{(0,1)}_b \circ P^{(0,0)}_d$ relative to the Haar basis $\{h_I\}_{I \in \mathcal{D}}$. 

A simple computation shows its entries are:

$$G_{I,J} = P^{(0,1)}_b \circ P^{(0,0)}_d \langle h_J, h_I \rangle_{L^2} = P^{(1,0)}_b \langle h_J, P^{(0,0)}_d h_I \rangle_{L^2} = d_{1(I)} \delta_{J,I} = \begin{cases} b_{I}d_{J} - \frac{1}{\sqrt{|J|}} & \text{if } I \subset J \\ b_{I}d_{J} + \frac{1}{\sqrt{|J|}} & \text{if } I \subset J \\ 0 & \text{if } J \subset I \text{ or } I \cap J = \emptyset \end{cases}.$$
The Gram Matrix of $P_b^{(0,1)} \circ P_d^{(0,0)}$

Let $G_{P_b^{(0,1)} \circ P_d^{(0,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ be the Gram matrix of the operator $P_b^{(0,1)} \circ P_d^{(0,0)}$ relative to the Haar basis $\{h_I\}_{I \in \mathcal{D}}$. A simple computation shows its entries are:

$$G_{I,J} = \left\langle P_b^{(0,1)} \circ P_d^{(0,0)} h_J, h_I \right\rangle_{L^2}$$
The Gram Matrix of $P_{b}^{(0,1)} \circ P_{d}^{(0,0)}$

Let $G_{P_{b}^{(0,1)} \circ P_{d}^{(0,0)}} = [G_{I,J}]_{I,J \in D}$ be the Gram matrix of the operator $P_{b}^{(0,1)} \circ P_{d}^{(0,0)}$ relative to the Haar basis $\{h_{I}\}_{I \in D}$. A simple computation shows its entries are:

$$G_{I,J} = \left\langle P_{b}^{(0,1)} \circ P_{d}^{(0,0)} h_{J}, h_{I} \right\rangle_{L^2} = \left\langle P_{d}^{(0,0)} h_{J}, P_{b}^{(1,0)} h_{I} \right\rangle_{L^2}.$$
The Gram Matrix of $P_b^{(0,1)} \circ P_d^{(0,0)}$

Let $\mathcal{G}_{P_b^{(0,1)} \circ P_d^{(0,0)}} = [G_I,J]_{I,J \in \mathcal{D}}$ be the Gram matrix of the operator $P_b^{(0,1)} \circ P_d^{(0,0)}$ relative to the Haar basis $\{h_I\}_{I \in \mathcal{D}}$. A simple computation shows its entries are:

$$G_{I,J} = \langle P_b^{(0,1)} \circ P_d^{(0,0)} h_J, h_I \rangle_{L^2} = \langle P_d^{(0,0)} h_J, P_b^{(1,0)} h_I \rangle_{L^2}$$

$$= \langle d_J h_J, b_I h_I^1 \rangle_{L^2}$$
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$$= \left\langle d_J h_J, b_I h_I^1 \right\rangle_{L^2}$$

$$= \overline{b_I} d_J h_I^1 (J)$$
The Gram Matrix of $P_{b}^{(0,1)} \circ P_{d}^{(0,0)}$

Let $G_{P_{b}^{(0,1)} \circ P_{d}^{(0,0)}} = [G_{I,J}]_{I,J \in D}$ be the Gram matrix of the operator $P_{b}^{(0,1)} \circ P_{d}^{(0,0)}$ relative to the Haar basis $\{h_{I}\}_{I \in D}$. A simple computation shows its entries are:

$$G_{I,J} = \langle P_{b}^{(0,1)} \circ P_{d}^{(0,0)} h_{J}, h_{I} \rangle_{L^{2}} = \langle P_{d}^{(0,0)} h_{J}, P_{b}^{(1,0)} h_{I} \rangle_{L^{2}} = \langle d_{J} h_{J}, b_{I} h_{I}^{1} \rangle_{L^{2}}$$

$$= \begin{cases} \overline{b_{I}} d_{J} \frac{-1}{\sqrt{|J|}} & \text{if } I \subset J_- \\ \overline{b_{I}} d_{J} \frac{1}{\sqrt{|J|}} & \text{if } I \subset J_+ \\ 0 & \text{if } J \subset I \text{ or } I \cap J = \emptyset. \end{cases}$$
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$$G_{I,J} = \left\langle P_{b}^{(0,1)} \circ P_{d}^{(0,0)} h_{J}, h_{I} \right\rangle_{L^{2}} = \left\langle P_{d}^{(0,0)} h_{J}, P_{b}^{(1,0)} h_{I} \right\rangle_{L^{2}} = \left\langle d_{J} h_{J}, b_{I} h_{I}^{1} \right\rangle_{L^{2}}$$

$$= \overline{b_{I}} d_{J} \left\langle h_{J}^{\dagger}, h_{I} \right\rangle_{L^{2}} = \begin{cases} \overline{b_{I}} d_{J} \left( \frac{-1}{\sqrt{|J|}} \right) & \text{if } I \subset J_{-} \\ \overline{b_{I}} d_{J} \left( \frac{1}{\sqrt{|J|}} \right) & \text{if } I \subset J_{+} \\ 0 & \text{if } J \subset I \text{ or } I \cap J = \emptyset. \end{cases}$$

**Idea:** Construct $T^{(0,1,0,0)}_{b,d} : L^{2}_{c}(\mathcal{H}) \rightarrow L^{2}_{c}(\mathcal{H})$ that has the same Gram matrix as $P_{b}^{(0,1)} \circ P_{d}^{(0,0)}$, but with respect to the basis $\{\widetilde{1}_{T(I)}\}_{I \in \mathcal{D}}$. 
The Operator $\mathcal{T}_{b,d}^{(0,1,0,0)}$

Now consider the operator $\mathcal{T}_{b,d}^{(0,1,0,0)}$ defined by

$$
\mathcal{T}_{b,d}^{(0,1,0,0)} \equiv \mathcal{M}_{\frac{1}{b}}^{-1} \left( \sum_{K \in \mathcal{D}} \tilde{1}_{Q_{\pm}(K)} \otimes \tilde{1}_{T(K)} \right) \mathcal{M}_{\frac{1}{d}}^{\frac{1}{2}}.
$$
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Now consider the operator $T_{b,d}^{(0,1,0,0)}$ defined by

$$T_{b,d}^{(0,1,0,0)} \equiv \mathcal{M}^{-1}_b \left( \sum_{K \in \mathcal{D}} \tilde{1}_{Q_\pm(K)} \otimes \tilde{1}_{T(K)} \right) \mathcal{M}_d^{1/2}.$$  

Here

$$1_{Q_\pm(K)} \equiv - \sum_{L \subset K_-} 1_{T(L)} + \sum_{L \subset K_+} 1_{T(L)}.$$
The Operator $T_{b,d}^{(0,1,0,0)}$

Now consider the operator $T_{b,d}^{(0,1,0,0)}$ defined by

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Here

$$1_{Q_{\pm}(K)} \equiv - \sum_{L \subset K_{-}} 1_{T(L)} + \sum_{L \subset K_{+}} 1_{T(L)}.$$

A straightforward computation shows

$$\left\| 1_{Q_{\pm}(K)} \right\|_{L^2(\mathcal{H})} = \frac{|K|}{2};$$
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Here

$$1_{Q_{\pm}(K)} \equiv - \sum_{L \subset K} 1_{T(L)} + \sum_{L \subset K} 1_{T(L)}.$$

A straightforward computation shows

$$\left\| 1_{Q_{\pm}(K)} \right\|_{L^{2}(\mathcal{H})} = \frac{|K|}{2};$$

$$\mathcal{M}_{a}^{\lambda} 1_{Q_{\pm}(K)} = - \sum_{L \subset K_{-}} a_{L} |L|^\lambda 1_{T(L)} + \sum_{L \subset K_{+}} a_{L} |L|^\lambda 1_{T(L)}.$$
The Gram Matrix for the Operator $T_{b,d}^{(0,1,0,0)}$

The Gram matrix $G_{T_{b,d}^{(0,1,0,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ of $T_{b,d}^{(0,1,0,0)}$ relative to the basis $\{\tilde{1}_{T(I)}\}_{I \in \mathcal{D}}$ then has entries given by

$$G_{I,J} = \left\langle T_{b,d}^{(0,1,0,0)} \tilde{1}_{T(J)} , \tilde{1}_{T(I)} \right\rangle_{L^2(\mathcal{H})}$$
The Gram Matrix for the Operator $T_{b,d}^{(0,1,0,0)}$

The Gram matrix $\mathcal{G}_{T_{b,d}^{(0,1,0,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ of $T_{b,d}^{(0,1,0,0)}$ relative to the basis $\{\tilde{1}_{T(I)}\}_{I \in \mathcal{D}}$ then has entries given by

$$G_{I,J} = \left\langle T_{b,d}^{(0,1,0,0)} \tilde{1}_{T(J)}, \tilde{1}_{T(I)} \right\rangle_{L^2(\mathcal{H})}$$

$$= \sqrt{2} \begin{cases} 
    -\overline{b_I} d_J |J|^{-1/2} & \text{if} \quad I \subset J_-
    \\
    \overline{b_I} d_J |J|^{-1/2} & \text{if} \quad I \subset J_+
    \\
    0 & \text{if} \quad J \subset I \text{ or } I \cap J = \emptyset.
\end{cases}$$
The Gram Matrix for the Operator $T_{b,d}^{(0,1,0,0)}$

The Gram matrix $\mathfrak{G}_{T_{b,d}^{(0,1,0,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ of $T_{b,d}^{(0,1,0,0)}$ relative to the basis $\{\widetilde{1}_{T(I)}\}_{I \in \mathcal{D}}$ then has entries given by

$$G_{I,J} = \left\langle T_{b,d}^{(0,1,0,0)} \widetilde{1}_{T(J)}, \widetilde{1}_{T(I)} \right\rangle_{L^2(\mathcal{H})}$$

$$= \sqrt{2} \begin{cases} -\overline{b}_I d_J |J|^{-\frac{1}{2}} & \text{if} \quad I \subset J_- \\ \overline{b}_I d_J |J|^{-\frac{1}{2}} & \text{if} \quad I \subset J_+ \\ 0 & \text{if} \quad J \subset I \text{ or } I \cap J = \emptyset. \end{cases}$$

Thus, up to an absolute constant, $\mathfrak{G}_{T_{b,d}^{(0,1,0,0)}} = \mathfrak{G}_{P_{b}^{(0,1)} \circ P_{d}^{(0,0)}}$, and so

$$\|P_{b}^{(0,1)} \circ P_{d}^{(0,0)}\|_{L^2 \to L^2} \approx \|T_{b,d}^{(0,1,0,0)}\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})}.$$
Connecting to a Two Weight Inequality

The inequality we wish to characterize is:

\[
\left\| M_b^{-1} U M_d^{\frac{1}{2}} f \right\|_{L_c^2(\mathcal{H})} = \left\| T_{b,d}^{(0,1,0,0)} f \right\|_{L_c^2(\mathcal{H})} \lesssim \| f \|_{L_c^2(\mathcal{H})}.
\]
Connecting to a Two Weight Inequality

The inequality we wish to characterize is:

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Where the operator $\mathcal{U}$ on $L^2(\mathcal{H})$ is defined by

$$\mathcal{U} \equiv \sum_{K \in \mathcal{D}} \tilde{1}_{Q_{\pm}(K)} \bigotimes \tilde{1}_{T(K)}.$$
Connecting to a Two Weight Inequality

The inequality we wish to characterize is:

\[ \left\| \mathcal{M}_{-b}^{-1} U \mathcal{M}_{d}^{1/2} f \right\|_{L^2_c(\mathcal{H})} = \left\| T_{b,d}^{(0,1,0,0)} f \right\|_{L^2_c(\mathcal{H})} \lesssim \|f\|_{L^2_c(\mathcal{H})}. \]

Where the operator \( U \) on \( L^2(\mathcal{H}) \) is defined by

\[ U \equiv \sum_{K \in \mathcal{D}} \tilde{1}_{Q_\pm(K)} \otimes \tilde{1}_{T(K)}. \]

One sees that the inequality to be characterized is equivalent to:

\[ \| U (\mu g) \|_{L^2_c(\mathcal{H};\nu)} \lesssim \|g\|_{L^2_c(\mathcal{H};\mu)}, \]

where the weights \( \mu \) and \( \nu \) are given by

\[ \nu \equiv \sum_{I \in \mathcal{D}} |b_I|^2 |I|^{-2} 1_{T(I)}, \]

\[ \mu \equiv \sum_{I \in \mathcal{D}} |d_I|^{-2} |I|^{-1} 1_{T(I)}. \]
Proofs of Main Results

The Characterization of Type \((0, 1, 0, 0)\)

Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

Suppose that \(\mu\) and \(\nu\) are positive measures on \(H\) that are constant on tiles, i.e.,

\[
\mu \equiv \sum_{I \in D} \mu_I 1_{T(I)},
\]

\[
\nu \equiv \sum_{I \in D} \nu_I 1_{T(I)}.
\]

Then

\[
U(\mu \cdot \cdot) : L^2_c(H; \mu) \to L^2_c(H; \nu)
\]

if and only if both

\[
\|U(\mu 1_{T(I)})\|_{L^2_c(H; \nu)} \leq C_1 \|1_{T(I)}\|_{L^2_c(H; \mu)} = \sum_{T(I)} \mu(T(I)),
\]

\[
\|1_{Q(I)} U(\nu 1_{Q(I)})\|_{L^2_c(H; \mu)} \leq C_2 \|1_{Q(I)}\|_{L^2_c(H; \nu)} = \sum_{Q(I)} \nu(Q(I)).
\]

hold for all \(I \in D\).

Moreover, we have that

\[
\|U\|_{L^2_c(H; \mu) \to L^2_c(H; \nu)} \approx C_1 + C_2
\]

where \(C_1\) and \(C_2\) are the best constants appearing above.
Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

Suppose that $\mu$ and $\nu$ are positive measures on $H$ that are constant on tiles, i.e., $\mu \equiv \sum_{I \in D} \mu_I 1_{T(I)}$, $\nu \equiv \sum_{I \in D} \nu_I 1_{T(I)}$. 

Moreover, we have that $\|U\|_{L^2_c(H; \mu)} \rightarrow L^2_c(H; \nu) \approx C_1 + C_2$ where $C_1$ and $C_2$ are the best constants appearing above.
Suppose that $\mu$ and $\nu$ are positive measures on $\mathcal{H}$ that are constant on tiles, i.e., $\mu \equiv \sum_{I \in \mathcal{D}} \mu_I 1_{T(I)}$, $\nu \equiv \sum_{I \in \mathcal{D}} \nu_I 1_{T(I)}$. Then

$$U(\mu \cdot) : L^2_c(\mathcal{H}; \mu) \rightarrow L^2_c(\mathcal{H}; \nu)$$

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$$\|U(\mu 1_{T(I)})\|_{L^2_c(\mathcal{H}; \nu)} \leq C_1 \|1_{T(I)}\|_{L^2_c(\mathcal{H}; \mu)} = \sqrt{\mu(T(I))},$$

$$\|1_{Q(I)} U^* (\nu 1_{Q(I)})\|_{L^2_c(\mathcal{H}; \mu)} \leq C_2 \|1_{Q(I)}\|_{L^2_c(\mathcal{H}; \nu)} = \sqrt{\nu(Q(I))},$$

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An Application: Linear Bound for Hilbert Transform

- For a weight \( w \), i.e., a positive locally integrable function on \( \mathbb{R} \), let \( L^2(w) \equiv L^2(\mathbb{R}; w) \).
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- However, each term can be shown to have norm no worse than $[w]_{A_2}$. 

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An Open Question

Unfortunately, the methods described do not appear to work to handle type \((0, 1, 0, 1)\) compositions. However, the following question is of interest:

For each \(I \in D\) determine function \(F_I, B_I \in L^2\) of norm 1 such that

\[
P(0, 1) \circ P(0, 1) \text{ is bounded on } L^2 \iff \|P(0, 1) \circ P(0, 1)\|_{L^2} \leq C_1 \forall I \in D;
\]

\[
P(1, 0) \circ P(1, 0) \text{ is bounded on } L^2 \iff \|P(1, 0) \circ P(1, 0)\|_{L^2} \leq C_2 \forall I \in D.
\]

Moreover, we will have

\[
\|P(0, 1) \circ P(0, 1)\|_{L^2} \to L^2 \approx C_1 + C_2.
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**Question**

For each $I \in \mathcal{D}$ determine function $F_I, B_I \in L^2$ of norm 1 such that $P_b^{(0,1)} \circ P_d^{(0,1)}$ is bounded on $L^2$ if and only if

$$\left\| P_b^{(0,1)} \circ P_d^{(0,1)} F_I \right\|_{L^2} \leq C_1 \quad \forall I \in \mathcal{D};$$

$$\left\| P_d^{(1,0)} \circ P_b^{(1,0)} B_I \right\|_{L^2} \leq C_2 \quad \forall I \in \mathcal{D}.$$

Moreover, we will have

$$\left\| P_b^{(0,1)} \circ P_d^{(0,1)} \right\|_{L^2 \rightarrow L^2} \approx C_1 + C_2.$$
Thanks to Nicola, John, Marco, Stefan, and Maura for Organizing the Meeting!
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The daydreams of cat herders
(Modified from the Original Dr. Fun Comic)
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