Dual Hilbert function spaces and dual functional models of linear operators

Dmitry Yakubovich

Universidad Autónoma de Madrid
e Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Spain)

co-authors: José Galé and Pedro Miana (Univ. Zaragoza, Spain)

May 23, 2013

1. General ideas on the dual models
2. Examples of dual models
3. Generalities on Nagy-Foiaş type models
4. Application to generators of analytic semigroups
5. Other applications of the Nagy–Foiaş type models
6. A concluding remark

Almost-eigenvectors and the ultraspectrum

Let $A \in \mathcal{L}(X)$, X a Hilbert space.

Almost-eigenvectors and the ultraspectrum

Let $A \in \mathcal{L}(X)$, X a Hilbert space. Suppose

$$U \xrightarrow{B} X \xrightarrow{C} Y,$$

inputs outputs;

$$U \xleftarrow{B^*} X \xleftarrow{C^*} Y.$$

Almost-eigenvectors and the ultraspectrum

Let $A \in \mathcal{L}(X)$, X a Hilbert space. Suppose

$$U \xrightarrow{B} X \xrightarrow{C} Y,$$

inputs outputs;

$$U \xleftarrow{B^*} X \xleftarrow{C^*} Y.$$

Almost-eigenvectors of A^*:

$$\begin{aligned} (\bar{z}I - A^*)g_{z,u,y} &= C^*y, \\ B^*g_{z,u,y} &= u, \qquad (u,y) \in U \oplus Y. \end{aligned}$$

Almost-eigenvectors and the ultraspectrum

Let $A \in \mathcal{L}(X)$, X a Hilbert space. Suppose

$$U \xrightarrow{B} X \xrightarrow{C} Y,$$

inputs outputs;

$$U \xleftarrow{B^*} X \xleftarrow{C^*} Y.$$

Almost-eigenvectors of A^*:

$$(\bar{z}I - A^*)g_{z,u,y} = C^*y,$$
$$B^*g_{z,u,y} = u, \qquad (u,y) \in U \oplus Y.$$

Assumption: (z,u,y) determine $g = g_{z,u,y}$.

$$\exists g = g_{z,u,y} \iff (u,y) \in \mathcal{F}(z).$$

The ultraspectrum of A: the vector bundle

$$\mathcal{F} = \{\mathcal{F}(z)\}_{z\in\mathbb{C}}$$

(of varying dimension, in general).

Definition of $\mathcal{O}_{A,B,C}$.

$$x \in X \mapsto \mathcal{O}_{A,B,C}\, x, \quad \big(\mathcal{O}_{A,B,C}\, x\big)(z) = \big\langle x, g_{z,u,y} \big\rangle.$$

$$\mathcal{O}_{A,B,C} : X \to \{\text{holomorphic cross-sections of } \mathcal{F}^* = \{\mathcal{F}(z)^*\}_{z\in\mathbb{C}}\},$$

Definition of $\mathcal{O}_{A,B,C}$.

$$x \in X \mapsto \mathcal{O}_{A,B,C}\, x, \quad \big(\mathcal{O}_{A,B,C}\, x\big)(z) = \big\langle x, g_{z,u,y} \big\rangle.$$

$$\mathcal{O}_{A,B,C} : X \to \{\text{holomorphic cross-sections of } \mathcal{F}^* = \{\mathcal{F}(z)^*\}_{z \in \mathbb{C}}\},$$

$$\boxed{(\mathcal{O}_{A,B,C} A x)(z) = \big(M_z^T \mathcal{O}_{A,B,C} x\big)(z),}$$

$$M_z^T f(z) \stackrel{\text{def}}{=} z f(z) - \big(z f(z)\big)\big|_{z=\infty}.$$

Definition of $\mathcal{O}_{A,B,C}$.

$$x \in X \mapsto \mathcal{O}_{A,B,C}\, x, \quad \big(\mathcal{O}_{A,B,C}\, x\big)(z) = \big\langle x, g_{z,u,y} \big\rangle.$$

$$\mathcal{O}_{A,B,C} : X \to \{\text{holomorphic cross-sections of } \mathcal{F}^* = \{\mathcal{F}(z)^*\}_{z\in\mathbb{C}}\},$$

$$\boxed{(\mathcal{O}_{A,B,C} A x)(z) = \big(M_z^T \mathcal{O}_{A,B,C} x\big)(z),}$$

$$M_z^T f(z) \stackrel{\text{def}}{=} z f(z) - \big(z f(z)\big)\big|_{z=\infty}.$$

General scheme:

1) Find almost-eigenvectors of A and of A^*.
2) Define model spaces $\mathrm{Mod}(\mathcal{F})$, $\mathrm{Mod}(\mathcal{F}_*)$ for A and A^*.
3) Define the Cauchy duality between $\mathrm{Mod}(\mathcal{F})$ and $\mathrm{Mod}(\mathcal{F}_*)$
4) Prove **the Model Theorem:**

Definition of $\mathcal{O}_{A,B,C}$.

$$x \in X \mapsto \mathcal{O}_{A,B,C}\, x, \quad \big(\mathcal{O}_{A,B,C}\, x\big)(z) = \big\langle x, g_{z,u,y} \big\rangle.$$

$$\mathcal{O}_{A,B,C} : X \to \{\text{holomorphic cross-sections of } \mathcal{F}^* = \{\mathcal{F}(z)^*\}_{z\in\mathbb{C}}\},$$

$$\boxed{(\mathcal{O}_{A,B,C} A x)(z) = \big(M_z^T \mathcal{O}_{A,B,C} x\big)(z),}$$

$$M_z^T f(z) \stackrel{\text{def}}{=} z f(z) - \big(z f(z)\big)\big|_{z=\infty}.$$

General scheme:

1) Find almost-eigenvectors of A and of A^*.
2) Define model spaces $\mathrm{Mod}(\mathcal{F})$, $\mathrm{Mod}(\mathcal{F}_*)$ for A and A^*.
3) Define the Cauchy duality between $\mathrm{Mod}(\mathcal{F})$ and $\mathrm{Mod}(\mathcal{F}_*)$
4) Prove **the Model Theorem:** A is similar to the operator of multiplication by the independent variable on the model space.

Basic idea: to construct simultaneously the models of A and A^*.

$$\mathcal{O} = \mathcal{O}_{A,B,C}; \qquad \mathcal{O}_* = \mathcal{O}_{A^*,C^*,B^*}.$$

$$\begin{array}{cc} X & X^* \\ \mathcal{O}\downarrow & \mathcal{O}_*\downarrow \\ \mathrm{Mod}(\mathcal{F}) & \mathrm{Mod}(\mathcal{F}_*). \end{array}$$

$$f\ , \qquad g \quad \longmapsto \quad \langle f, g\rangle_d \in \mathbb{C}$$

1. General ideas on the dual models

Basic idea: to construct simultaneously the models of A and A^*.

$$\mathcal{O} = \mathcal{O}_{A,B,C}; \qquad \mathcal{O}_* = \mathcal{O}_{A^*,C^*,B^*}.$$

$$\begin{array}{cc} X & X^* \\ \mathcal{O}\downarrow & \mathcal{O}_*\downarrow \\ \mathrm{Mod}(\mathcal{F}) & \mathrm{Mod}(\mathcal{F}_*). \end{array}$$

$$f\ , \qquad g \quad \longmapsto \quad \langle f, g\rangle_d \in \mathbb{C}$$

Fact: $\mathrm{Mod}(\mathcal{F}) = \big(\mathrm{Mod}(\mathcal{F}_*)\big)^*$, $\mathcal{O}$, $\mathcal{O}_*$ bounded, one-to-one $\Longrightarrow$

they are isomorphisms,

$$\langle x, y\rangle = \langle \mathcal{O}x,\ \mathcal{O}_*y\rangle_d \qquad \forall x \in X \quad \forall y \in X^*.$$

Remarks 1. A simple calculation shows:

$$\langle g_{z,u,y},\ g_{*,w,u_*,y_*}\rangle = \frac{\langle u, u_*\rangle - \langle y, y_*\rangle}{w-z}, \quad (u,y)\in\mathcal{F}(z),\ (u_*,y_*)\in\mathcal{F}_*(w).$$

Remarks 1. A simple calculation shows:

$$\langle g_{z,u,y},\ g_{*,w,u_*,y_*}\rangle = \frac{\langle u, u_*\rangle - \langle y, y_*\rangle}{w-z}, \quad (u,y)\in\mathcal{F}(z),\ (u_*,y_*)\in\mathcal{F}_*(w).$$

It follows that the duality between $\mathrm{Mod}(\mathcal{F})$ and $\mathrm{Mod}(\mathcal{F}_*)$ is always a Cauchy duality.

1. General ideas on the dual models

Remarks 1. A simple calculation shows:

$$\langle g_{z,u,y},\ g_{*,w,u_*,y_*}\rangle = \frac{\langle u, u_*\rangle - \langle y, y_*\rangle}{w-z}, \quad (u,y)\in\mathcal{F}(z),\ (u_*,y_*)\in\mathcal{F}_*(w).$$

It follows that the duality between $\mathrm{Mod}(\mathcal{F})$ and $\mathrm{Mod}(\mathcal{F}_*)$ is always a Cauchy duality.

2. For $z\notin\sigma(A)$,

$$(u,y)\in\mathcal{F}(z) \iff u = \Phi(z)^* y,$$

where

$$\Phi(z) = C(z-A)^{-1}B : \quad U\to Y$$

is the transfer function of system (A,B,C). In this case, instead of values $\mathcal{O}_{A,B,C}x(z)$ we can consider the values of

$$\mathcal{O}_{A,C}x(z) \stackrel{\text{def}}{=} C(z-A)^{-1}x.$$

A. Toeplitz operators. $F \in C^{1+\epsilon}(\mathbb{T})$, $\epsilon > 0$; $F(\mathbb{T})$ has a finite number of self-intersections.

$$T_F a = P_{H^2}(F \cdot a), \qquad a \in H^2.$$

Case of positive winding:

$$B = P_{\mathrm{Lin}(1, z, z^2 \ldots, z^{N-1})}, \qquad C = 0.$$

A. Toeplitz operators. $F \in C^{1+\epsilon}(\mathbb{T})$, $\epsilon > 0$; $F(\mathbb{T})$ has a finite number of self-intersections.

$$T_F a = P_{H^2}(F \cdot a), \qquad a \in H^2.$$

Case of positive winding:

$$B = P_{\mathrm{Lin}(1,z,z^2\dots,z^{N-1})}, \qquad C = 0.$$

B. "Smooth" perturbations of normal operators with two-dimensional Lebesgue spectrum.

N normal; $\qquad$ $A = N + K$ its perturbation.

$$K = CB; \qquad \sigma(A) \subset \mathcal{D}.$$

Conclusion: $N + K$ is similar to the operator of multiplication by the independent variable on

$$\mathcal{H}_{\mathcal{D}} / \Theta E^2(\mathcal{D}).$$

C. Hyponormal operators.

A **hyponormal** operator T:

$$D \stackrel{\text{def}}{=} T^*T - TT^* \geq 0 :$$

Dual analytic models of T and of T^* in
spaces of distributions

D. Xia et al 1983, M. Martin, M. Putinar 1987, [Y] 1995.

$$B = D^{1/2}, \qquad C = 0.$$

D. Subnormal operators.

A **subnormal** operator S: $S = N|H$, $NH \subset H$, N normal: D. Xia 1987 – 1996, McCarthy-Yang 1997 et al.:

The case of finite rank self-commutator [Y], 1998:

Dual analytic models in *Hardy spaces over "halves" of real algebraic curves.*

3. Generalities on Nagy-Foiaş type models

Assume: $\sigma(A) \subset \Omega^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \operatorname{clos}\Omega$.
We wish to construct a Nagy-Foiaş type model of A in Ω.

Remind:

3. Generalities on Nagy-Foiaş type models

Assume: $\sigma(A) \subset \Omega^{\mathrm{cl}} \overset{\mathrm{def}}{=} \operatorname{clos}\Omega$.
We wish to construct a Nagy-Foiaș type model of A in Ω.

Remind:
The Hardy–Smirnov class

$$E^2(\Omega) \overset{\mathrm{def}}{=} \operatorname{clos}_{L^2(\partial\Omega)} \mathrm{Pol}$$

3. Generalities on Nagy-Foiaş type models

Assume: $\sigma(A) \subset \Omega^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \operatorname{clos}\Omega$.
We wish to construct a Nagy-Foiaș type model of A in Ω.

Remind:
The Hardy–Smirnov class

$$E^2(\Omega) \stackrel{\mathrm{def}}{=} \operatorname{clos}_{L^2(\partial\Omega)} \mathrm{Pol}$$

Vector-valued classes $E^2(\Omega, Y)$ are defined in the same way.

$\sigma(A) \subset \Omega^{\mathsf{cl}} \stackrel{\text{def}}{=} \operatorname{clos}\Omega.$

$$X \quad \underset{\mathcal{O}_{A,C}}{\longrightarrow} \quad \operatorname{Mod}(\mathcal{F}) \subset E_0^2(\mathbb{C} \setminus \Omega^{\mathsf{cl}}, Y)$$

3. Generalities on Nagy-Foiaş type models

$\sigma(A) \subset \Omega^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \operatorname{clos}\Omega.$

$$X \underset{\mathcal{O}_{A,C}}{\longrightarrow} \operatorname{Mod}(\mathcal{F}) \subset E_0^2(\mathbb{C} \setminus \Omega^{\mathrm{cl}}, Y)$$

$$X^* \underset{\mathcal{O}_{A^*,B^*}}{\longrightarrow} \operatorname{Mod}(\mathcal{F}_*) \subset E_0^2(\mathbb{C} \setminus \bar{\Omega}^{\mathrm{cl}}, U).$$

$\sigma(A) \subset \Omega^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \operatorname{clos}\Omega.$

$$E^2(\Omega, U) \underset{W_{A,B}}{\dashrightarrow} X \underset{\mathcal{O}_{A,C}}{\longrightarrow} \mathrm{Mod}(\mathcal{F}) \subset E^2_0(\mathbb{C} \setminus \Omega^{\mathrm{cl}}, Y)$$

$$X^* \underset{\mathcal{O}_{A^*,B^*}}{\longrightarrow} \mathrm{Mod}(\mathcal{F}_*) \subset E^2_0(\mathbb{C} \setminus \bar{\Omega}^{\mathrm{cl}}, U).$$

3. Generalities on Nagy-Foiaş type models

$\sigma(A) \subset \Omega^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \operatorname{clos}\Omega.$

$$
\begin{array}{ccccc}
E^2(\Omega, U) & \underset{W_{A,B}}{\dashrightarrow} & X & \underset{\mathcal{O}_{A,C}}{\longrightarrow} & \mathrm{Mod}(\mathcal{F}) \subset E_0^2(\mathbb{C} \setminus \Omega^{\mathrm{cl}}, Y) \\
 & & X^* & \underset{\mathcal{O}_{A^*,B^*}}{\longrightarrow} & \mathrm{Mod}(\mathcal{F}_*) \subset E_0^2(\mathbb{C} \setminus \overline{\Omega}^{\mathrm{cl}}, U).
\end{array}
$$

Here

$$W_{A,B} = \left(\mathcal{O}_{A^*,B^*}\right)^* : E^2(\Omega, U) \to H.$$

$\sigma(A) \subset \Omega^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \operatorname{clos}\Omega.$

$$
\begin{array}{ccccc}
E^2(\Omega, U) & \underset{W_{A,B}}{\dashrightarrow} & X & \underset{\mathcal{O}_{A,C}}{\longrightarrow} & \mathrm{Mod}(\mathcal{F}) \subset E_0^2(\mathbb{C} \setminus \Omega^{\mathrm{cl}}, Y) \\
 & & X^* & \underset{\mathcal{O}_{A^*,B^*}}{\longrightarrow} & \mathrm{Mod}(\mathcal{F}_*) \subset E_0^2(\mathbb{C} \setminus \bar{\Omega}^{\mathrm{cl}}, U).
\end{array}
$$

Here

$$W_{A,B} = \left(\mathcal{O}_{A^*,B^*}\right)^* : E^2(\Omega, U) \to H.$$

$W_{A,B}$ is defined on U-valued rational functions by

$$W_{A,B}\big(q(z)u\big) \stackrel{\mathrm{def}}{=} q(A)Bu \in X, \qquad u \in U, q \in \mathrm{Rat}(\Omega^{\mathrm{cl}}).$$

$\sigma(A) \subset \Omega^{\mathrm{cl}} \stackrel{\text{def}}{=} \operatorname{clos}\Omega.$

$$\begin{array}{ccccc} E^2(\Omega, U) & \underset{W_{A,B}}{\dashrightarrow} & X & \xrightarrow[\mathcal{O}_{A,C}]{} & \mathrm{Mod}(\mathcal{F}) \subset E^2_0(\mathbb{C} \setminus \Omega^{\mathrm{cl}}, Y) \\ & & X^* & \xrightarrow[\mathcal{O}_{A^*,B^*}]{} & \mathrm{Mod}(\mathcal{F}_*) \subset E^2_0(\mathbb{C} \setminus \bar{\Omega}^{\mathrm{cl}}, U). \end{array}$$

Here

$$W_{A,B} = \left(\mathcal{O}_{A^*,B^*}\right)^* : E^2(\Omega, U) \to H.$$

$W_{A,B}$ is defined on U-valued rational functions by

$$W_{A,B}\big(q(z)u\big) \stackrel{\text{def}}{=} q(A)Bu \in X, \qquad u \in U, q \in \mathrm{Rat}(\Omega^{\mathrm{cl}}).$$

Then $W_{A,B}$ is extended by continuity to $E^2(\Omega, U)$.

$$W_{A,B} M_z = A W_{A,B}.$$

3. Generalities on Nagy-Foiaş type models

$\sigma(A) \subset \Omega^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \operatorname{clos}\Omega.$

$$E^2(\Omega, U) \quad \underset{W_{A,B}}{\dashrightarrow} \quad X \quad \underset{\mathcal{O}_{A,C}}{\longrightarrow} \quad \mathrm{Mod}(\mathcal{F}) \subset E_0^2(\mathbb{C} \setminus \Omega^{\mathrm{cl}}, Y)$$

Here

$$W_{A,B} = \left(\mathcal{O}_{A^*,B^*}\right)^* : E^2(\Omega, U) \to H.$$

$W_{A,B}$ is defined on U-valued rational functions by

$$W_{A,B}\big(q(z)u\big) \stackrel{\mathrm{def}}{=} q(A)Bu \in X, \qquad u \in U, q \in \mathrm{Rat}(\Omega^{\mathrm{cl}}).$$

Then $W_{A,B}$ is extended by continuity to $E^2(\Omega, U)$.

$$W_{A,B} M_z = A W_{A,B}.$$

3. Generalities on Nagy-Foiaş type models

$$B : U \to X, \qquad W_{A,B} : E^2(\Omega, U) \to X$$

3. Generalities on Nagy-Foiaş type models

$$B : U \to X, \qquad W_{A,B} : E^2(\Omega, U) \to X$$

Definition

(A, B) is exact

3. Generalities on Nagy-Foiaş type models

$$B : U \to X, \qquad W_{A,B} : E^2(\Omega, U) \to X$$

Definition

(A, B) is exact if $W_{A,B}$ is continuous on $E^2(\Omega, U)$ and surjective.

$$B : U \to X, \qquad W_{A,B} : E^2(\Omega, U) \to X$$

Definition

(A, B) is exact if $W_{A,B}$ is continuous on $E^2(\Omega, U)$ and surjective.

Definition

$\ker W_{A,B} \stackrel{\text{def}}{=} \delta E^2(\Omega, Y)$;

3. Generalities on Nagy-Foiaş type models

$$B : U \to X, \qquad\qquad W_{A,B} : E^2(\Omega, U) \to X$$

Definition

(A, B) is exact if $W_{A,B}$ is continuous on $E^2(\Omega, U)$ and surjective.

Definition

$\ker W_{A,B} \stackrel{\text{def}}{=} \delta E^2(\Omega, Y)$;
$\delta \in H^\infty\big(\Omega, Y \to U\big)$ is a generalized characteristic function of (A, B).

3. Generalities on Nagy-Foiaş type models

$$B : U \to X, \qquad W_{A,B} : E^2(\Omega, U) \to X$$

Definition

(A, B) is exact if $W_{A,B}$ is continuous on $E^2(\Omega, U)$ and surjective.

Definition

$\ker W_{A,B} \stackrel{\text{def}}{=} \delta E^2(\Omega, Y)$;
$\delta \in H^\infty\big(\Omega, Y \to U\big)$ is a generalized characteristic function of (A, B).

Note: Y is the space of outputs, $\delta(z) : Y \to U$.

3. Generalities on Nagy-Foiaş type models

$$B : U \to X, \qquad\qquad W_{A,B} : E^2(\Omega, U) \to X$$

Definition

(A, B) is exact if $W_{A,B}$ is continuous on $E^2(\Omega, U)$ and surjective.

Definition

$\ker W_{A,B} \stackrel{\mathrm{def}}{=} \delta E^2(\Omega, Y)$;
$\delta \in H^\infty\big(\Omega, Y \to U\big)$ is a generalized characteristic function of (A, B).

Note: Y is the space of outputs, $\delta(z) : Y \to U$.

We could have assumed that δ is an inner function.

3. Generalities on Nagy-Foiaş type models

$$B : U \to X, \qquad\qquad W_{A,B} : E^2(\Omega, U) \to X$$

Definition

(A, B) is exact if $W_{A,B}$ is continuous on $E^2(\Omega, U)$ and surjective.

Definition

$\ker W_{A,B} \stackrel{\text{def}}{=} \delta E^2(\Omega, Y)$;
$\delta \in H^\infty\big(\Omega, Y \to U\big)$ is a generalized characteristic function of (A, B).

Note: Y is the space of outputs, $\delta(z) : Y \to U$.

We could have assumed that δ is an inner function.

However, we assume only $\|\delta^{-1}|\partial\Omega\| \le C$.

Definition of $\widehat{W}_{A,B}$:

Definition of $\widehat{W}_{A,B}$:

$$E^2(\Omega, U) \xrightarrow{\hspace{6cm}} X$$

Definition of $\widehat{W}_{A,B}$:

$$E^2(\Omega, U) \xrightarrow{\qquad W_{A,B} \qquad} X$$

Definition of $\widehat{W}_{A,B}$:

$$E^2(\Omega, U)/\delta E^2(\Omega, Y) \xrightarrow{\quad W_{A,B} \quad} X$$

Definition of $\widehat{W}_{A,B}$:

$$E^2(\Omega, U)/\delta E^2(\Omega, Y) \xrightarrow{\quad \widehat{W}_{A,B} \quad} X$$

3. Generalities on Nagy-Foiaş type models

Definition of $\widehat{W}_{A,B}$:

$$E^2(\Omega, U)/\delta E^2(\Omega, Y) \xrightarrow{\widehat{W}_{A,B}} X$$

Theorem **(control model).** Suppose system (A, B) is exactly controllable, and define its G.Ch.F. θ and a space Y by $\operatorname{Ker} W = \delta E^2(\Omega, Y)$. Then A is similar to the operator of multiplication by the independent variable on

$$E^2(\Omega, U)/\delta E^2(\Omega, Y).$$

The converse also holds true.

Definition of $\widehat{W}_{A,B}$:

$$E^2(\Omega, U)/\delta E^2(\Omega, Y) \xrightarrow{\widehat{W}_{A,B}} X$$

Theorem (control model). Suppose system (A, B) is exactly controllable, and define its G.Ch.F. θ and a space Y by $\operatorname{Ker} W = \delta E^2(\Omega, Y)$. Then A is similar to the operator of multiplication by the independent variable on

$$E^2(\Omega, U)/\delta E^2(\Omega, Y).$$

The converse also holds true.

One of the main questions

How to calculate δ explicitly ?

3. Generalities on Nagy-Foiaş type models

The duality between the observation and the control models:

The duality between the observation and the control models:

$$E^2(\Omega, U)/\delta E^2(\Omega, Y) \xrightarrow{\widehat{W_{A,B}}} X \xrightarrow{\mathcal{O}_{A,C}} \mathrm{Mod}(\mathcal{F}) = \mathcal{H}(\delta)$$

The duality between the observation and the control models:

$$\begin{tikzcd} E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "J_\delta"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathrm{Mod}(\mathcal{F}) = \mathcal{H}(\delta) \\ & X \arrow[ur, "\mathcal{O}_{A,C}"'] & \end{tikzcd}$$

The duality between the observation and the control models:

$$\begin{tikzcd}
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "J_\delta"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathrm{Mod}(\mathcal{F}) = \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}$$

The duality between the observation and the control models:

$$\begin{tikzcd}
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "J_\delta"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathrm{Mod}(\mathcal{F}) = \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}$$

Definition

$$J_\delta : E^2(\Omega, U) \to E^2(\mathbb{C} \setminus \bar{\Omega}, Y), \qquad J_\delta f \overset{\text{def}}{=} P_-\left(\delta^{-1} f\right), \quad f \in E^2(\Omega, U).$$

where
$$L^2(\partial\Omega) = \underset{P_+}{E^2(\Omega)} \dot{+} \underset{P_-}{E^2(\mathbb{C} \setminus \Omega^{\mathrm{cl}})}$$

3. Generalities on Nagy-Foiaş type models

The duality between the observation and the control models:

$$
\begin{tikzcd}
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "J_\delta"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathrm{Mod}(\mathcal{F}) = \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}
$$

Definition

$$J_\delta : E^2(\Omega, U) \to E^2(\mathbb{C} \setminus \bar{\Omega}, Y), \qquad J_\delta f \overset{\text{def}}{=} P_-\big(\delta^{-1} f\big), \quad f \in E^2(\Omega, U).$$

where
$$L^2(\partial\Omega) = \underset{P_+}{E^2(\Omega)} \dotplus \underset{P_-}{E^2(\mathbb{C} \setminus \Omega^{\mathrm{cl}})}$$

$$\text{Range}\, J_\delta \overset{\text{def}}{=} \mathcal{H}(\delta) = \big\{ g \in E^2(\mathbb{C} \setminus \bar{\Omega}, Y) : \quad \delta \cdot g|\partial\Omega \in E^2(\Omega, U) \big\}.$$

The duality between the observation and the control models:

$$\begin{tikzcd} E^2(\Omega,U)/\delta E^2(\Omega,Y) \arrow[rr, "\widehat{J_\delta}"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathrm{Mod}(\mathcal{F}) = \mathcal{H}(\delta) \\ & X \arrow[ur, "\mathcal{O}_{A,C}"'] & \end{tikzcd}$$

Definition

$$J_\delta : E^2(\Omega,U) \to E^2(\mathbb{C}\setminus\bar{\Omega},Y), \qquad J_\delta f \overset{\text{def}}{=} P_-\left(\delta^{-1}f\right), \quad f \in E^2(\Omega,U).$$

where $$L^2(\partial\Omega) = \underset{P_+}{E^2(\Omega)} \dot{+} \underset{P_-}{E^2(\mathbb{C}\setminus\Omega^{\mathrm{cl}})}$$

$$\text{Range}\, J_\delta \overset{\text{def}}{=} \mathcal{H}(\delta) = \left\{g \in E^2(\mathbb{C}\setminus\bar{\Omega},Y) : \quad \delta\cdot g|\partial\Omega \in E^2(\Omega,U)\right\}.$$

3. Generalities on Nagy-Foiaş type models

$$\begin{tikzcd}
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "\widehat{J_\delta}"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "\widehat{J_\delta}"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}$$

Main technical result

Suppose $W_{A,B}, \mathcal{O}_{A,C}$ are bounded, $\mathcal{O}_{A,C}$ is one to one and $W_{A,B}$ has dense range. Then

$$\begin{tikzcd}
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "\widehat{J_\delta}"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}$$

Main technical result

Suppose $W_{A,B}, \mathcal{O}_{A,C}$ are bounded, $\mathcal{O}_{A,C}$ is one to one and $W_{A,B}$ has dense range. Then $\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J_\delta}$ $\Rightarrow$

3. Generalities on Nagy-Foiaş type models

$$
\begin{tikzcd}
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "\widehat{J}_\delta"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}
$$

Main technical result

Suppose $W_{A,B}, \mathcal{O}_{A,C}$ are bounded, $\mathcal{O}_{A,C}$ is one to one and $W_{A,B}$ has dense range. Then $\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J}_\delta \;\Rightarrow$ both $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms, and we obtain both control and observation Nagy-Foiaș type models of A in Ω.

$$\begin{tikzcd}
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "\widehat{J_\delta}"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}$$

Main technical result

Suppose $W_{A,B}, \mathcal{O}_{A,C}$ are bounded, $\mathcal{O}_{A,C}$ is one to one and $W_{A,B}$ has dense range. Then $\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J_\delta} \;\; \Rightarrow$
both $\mathcal{O}_{A,C}$ and $\widehat{W_{A,B}}$ are isomorphisms, and we obtain both control and observation Nagy-Foiaş type models of A in Ω.

Remark.

$$
\begin{tikzcd}
E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "\widehat{J}_\delta"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathcal{H}(\delta) \\
& X \arrow[ur, "\mathcal{O}_{A,C}"'] &
\end{tikzcd}
$$

Main technical result

Suppose $W_{A,B}, \mathcal{O}_{A,C}$ are bounded, $\mathcal{O}_{A,C}$ is one to one and $W_{A,B}$ has dense range. Then $\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J}_\delta \quad \Rightarrow$
both $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms, and we obtain both control and observation Nagy-Foiaș type models of A in Ω.

Remark.

$\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J}_\delta \quad \Longleftrightarrow$

3. Generalities on Nagy-Foiaş type models

$$\begin{tikzcd} E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "\widehat{J}_\delta"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathcal{H}(\delta) \\ & X \arrow[ur, "\mathcal{O}_{A,C}"'] & \end{tikzcd}$$

Main technical result

Suppose $W_{A,B}, \mathcal{O}_{A,C}$ are bounded, $\mathcal{O}_{A,C}$ is one to one and $W_{A,B}$ has dense range. Then $\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J}_\delta \ \Rightarrow$
both $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms, and we obtain both control and observation Nagy-Foiaș type models of A in Ω.

Remark.

$$\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J}_\delta \quad \Longleftrightarrow \quad \Phi u = P_-\left(\delta^{-1}u\right) \text{ on } \mathbb{C} \setminus \Omega^{\mathrm{cl}} \quad \forall u \in U$$

3. Generalities on Nagy-Foiaş type models

$$\begin{tikzcd} E^2(\Omega, U)/\delta E^2(\Omega, Y) \arrow[rr, "\widehat{J}_\delta"] \arrow[dr, "\widehat{W_{A,B}}"'] & & \mathcal{H}(\delta) \\ & X \arrow[ur, "\mathcal{O}_{A,C}"'] & \end{tikzcd}$$

Main technical result

Suppose $W_{A,B}, \mathcal{O}_{A,C}$ are bounded, $\mathcal{O}_{A,C}$ is one to one and $W_{A,B}$ has dense range. Then $\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J}_\delta \quad \Rightarrow$
both $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms, and we obtain both control and observation Nagy-Foiaș type models of A in Ω.

Remark.

$\mathcal{O}_{A,C}\widehat{W_{A,B}} = \widehat{J}_\delta \quad \Longleftrightarrow \quad \Phi u = P_-\left(\delta^{-1} u\right)$ on $\mathbb{C} \setminus \Omega^{\mathrm{cl}} \quad \forall u \in U$
where $\Phi(z) = C(z - A)^{-1}B : U \to Y$ is the transfer function.

4. Application to generators of analytic semigroups

Suppose $0 < \theta < \pi/2$,

4. Application to generators of analytic semigroups

Suppose $0 < \theta < \pi/2$,

$$\Omega = \Omega_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}.$$

4. Application to generators of analytic semigroups

Suppose $0 < \theta < \pi/2$,

$$\Omega = \Omega_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}.$$

Let A be a closed operator on a Hilbert space X,

4. Application to generators of analytic semigroups

Suppose $0 < \theta < \pi/2$,

$$\Omega = \Omega_\theta = \left\{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\right\}.$$

Let A be a closed operator on a Hilbert space X,

$$\|(z - A)^{-1}\| \leq \frac{C}{|z|}, \qquad z \notin \Omega_\theta.$$

We assume $\ker A = 0$.

Theorem (A. McIntosh, 1986)

Let $\Omega = \Omega_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}$ *and* A *is as above. Then TFAE:*

(*a*) *A admits an $H^\infty(\Omega)$ functional calculus;*

(*b*) $\{A^{is}\}_{s\in\mathbb{R}}$ *is a C_0 group;*

(*c*) $\forall\alpha \in (0,1), \quad \mathcal{D}(A^\alpha) = \mathcal{D}(|A|^\alpha)$, $\mathcal{D}(A^{*\alpha}) = \mathcal{D}(|A^*|^\alpha)$.

Theorem (A. McIntosh, 1986)

Let $\Omega = \Omega_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}$ *and A is as above. Then TFAE:*

(a) *A admits an $H^\infty(\Omega)$ functional calculus;*

(b) $\{A^{is}\}_{s\in\mathbb{R}}$ *is a C_0 group;*

(c) $\forall \alpha \in (0,1), \quad \mathcal{D}(A^\alpha) = \mathcal{D}(|A|^\alpha), \mathcal{D}(A^{*\alpha}) = \mathcal{D}(|A^*|^\alpha).$

By applying Nagy-Foiaș type models, we obtain the following

Theorem (J. Galé–P. Miana–Y.)

Put

$$U = Y = H, \qquad B = 2\sqrt{A}, \qquad B = \sqrt{A}.$$

Then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow$

Theorem (A. McIntosh, 1986)

Let $\Omega = \Omega_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}$ *and* A *is as above. Then TFAE:*

(a) *A admits an $H^\infty(\Omega)$ functional calculus;*

(b) $\{A^{is}\}_{s\in\mathbb{R}}$ *is a* C_0 *group;*

(c) $\forall \alpha \in (0,1), \quad \mathcal{D}(A^\alpha) = \mathcal{D}(|A|^\alpha), \mathcal{D}(A^{*\alpha}) = \mathcal{D}(|A^*|^\alpha)$.

By applying Nagy-Foiaș type models, we obtain the following

Theorem (J. Galé–P. Miana–Y.)

Put

$$U = Y = H, \qquad B = 2\sqrt{A}, \qquad B = \sqrt{A}.$$

Then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow$

(d) $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms, with

$$\delta(z) = \frac{A - z}{A + z}.$$

4. Application to generators of analytic semigroups

Remark: McIntosh's construction is related with:

- The Kato square root problem

4. Application to generators of analytic semigroups

Remark: McIntosh's construction is related with:

- The Kato square root problem
- maximal regularity

4. Application to generators of analytic semigroups

Remark: McIntosh's construction is related with:

- The Kato square root problem
- maximal regularity
- boundedness of the Cauchy integral operator on a Lipschitz curve

4. Application to generators of analytic semigroups

Remark: McIntosh's construction is related with:

- The Kato square root problem
- maximal regularity
- boundedness of the Cauchy integral operator on a Lipschitz curve
- The Banach space case is also very interesting.

5. Other applications of the Nagy–Foiaş type models

5. Other applications of the Nagy–Foiaş type models

- A = a first order non-dissipative differential operator,
  Ω = a left half-plane;
  δ will be an explicit $n \times n$ entire function;

5. Other applications of the Nagy–Foiaş type models

- A = a first order non-dissipative differential operator,
  Ω = a left half-plane;
  δ will be an explicit $n \times n$ entire function;
- A = a C_0 generator of a linear neutral system with delays,
  Ω = a left half-plane;
  δ will be an explicit $n \times n$ holomorphic function in $\mathbb{C}$
  (except for one first order pole);

5. Other applications of the Nagy–Foiaş type models

- A $=$ a first order non-dissipative differential operator,
  Ω $=$ a left half-plane;
  δ will be an explicit $n \times n$ entire function;
- A $=$ a C_0 generator of a linear neutral system with delays,
  Ω $=$ a left half-plane;
  δ will be an explicit $n \times n$ holomorphic function in $\mathbb{C}$
  (except for one first order pole);
- A $=$ a generator of an arbitrary C_0 group;
  Ω $=$ a suitable vertical strip;
  $\delta(z) = (z - A)(z + \sigma - A)^{-1}, \quad \sigma = \text{const};$

5. Other applications of the Nagy–Foiaş type models

- $A \; =$ a first order non-dissipative differential operator,
  $\Omega \; =$ a left half-plane;
  δ will be an explicit $n \times n$ entire function;
- $A \; =$ a C_0 generator of a linear neutral system with delays,
  $\Omega \; =$ a left half-plane;
  δ will be an explicit $n \times n$ holomorphic function in $\mathbb{C}$
  (except for one first order pole);
- $A \; =$ a generator of an arbitrary C_0 group;
  $\Omega \; =$ a suitable vertical strip;
  $\delta(z) = (z - A)(z + \sigma - A)^{-1}, \;\; \sigma = \text{const}$;
- $A \; = A_0 + |A_0|^\alpha K |A_0|^\alpha$ non-dissipative operator $(\alpha \leq \frac{1}{2})$,
  $\Omega \; =$ a suitable non-convex parabolic domain,
  also an explicit formula for δ.

6. A concluding remark

Thank you for your attention

6. A concluding remark

Thank you for your attention !

References

References

1. D.Y, St. Petersburg Mathematical Journal (2004);

References

1. D.Y, St. Petersburg Mathematical Journal (2004);

2. S. M. Verduyn Lunel, D.Y., A functional model approach to linear neutral functional-differential equations, Integral Eqs. Operator Theory (1997);

References

1. D.Y, St. Petersburg Mathematical Journal (2004);

2. S. M. Verduyn Lunel, D.Y., A functional model approach to linear neutral functional-differential equations, Integral Eqs. Operator Theory (1997);

3. D.Y, J. Operator Theory (2008);

References

1. D.Y, St. Petersburg Mathematical Journal (2004);

2. S. M. Verduyn Lunel, D.Y., A functional model approach to linear neutral functional-differential equations, Integral Eqs. Operator Theory (1997);

3. D.Y, J. Operator Theory (2008);

4. P. Miana, J. Galé, D.Y, Math. Ann., 2011;

References

1. D.Y, St. Petersburg Mathematical Journal (2004);

2. S. M. Verduyn Lunel, D.Y., A functional model approach to linear neutral functional-differential equations, Integral Eqs. Operator Theory (1997);

3. D.Y, J. Operator Theory (2008);

4. P. Miana, J. Galé, D.Y, Math. Ann., 2011;

5. D.Y, Revista Matemática Iberoamericana, 1998 (2 papers);

References

1. D.Y, St. Petersburg Mathematical Journal (2004);

2. S. M. Verduyn Lunel, D.Y., A functional model approach to linear neutral functional-differential equations, Integral Eqs. Operator Theory (1997);

3. D.Y, J. Operator Theory (2008);

4. P. Miana, J. Galé, D.Y, Math. Ann., 2011;

5. D.Y, Revista Matemática Iberoamericana, 1998 (2 papers);

6. D.Y, JFA, 1996.

ADDITIONAL OBSERVATIONS

REMARK

In general, $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms if one replaces H with H_A,

Additional observations

Remark

In general, $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms if one replaces H with H_A, where H_A is the completion of $A(I+A)^{-2}H$ w.r. to the McIntosh norm

$$\|x\|_A := \left(\int_0^\infty \|\psi_t(A)x\|^2 \, \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $\psi \in H^\infty(\Omega_\theta)$, $\ |\psi(z)| \le \frac{|z|^s}{1+|z|^{2s}} \ \ (z \in \Omega_\theta)$, for some $s > 0$.

Additional observations

Remark

In general, $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms if one replaces H with H_A, where H_A is the completion of $A(I+A)^{-2}H$ w.r. to the McIntosh norm

$$\|x\|_A := \left(\int_0^\infty \|\psi_t(A)x\|^2 \, \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $\psi \in H^\infty(\Omega_\theta)$, $\quad |\psi(z)| \le \frac{|z|^s}{1+|z|^{2s}} \quad (z \in \Omega_\theta)$, for some $s > 0$.

In particular, there is an H^∞ calculus for A, considered as an operator on H_A.

Additional observations

Remark

In general, $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms if one replaces H with H_A, where H_A is the completion of $A(I+A)^{-2}H$ w.r. to the McIntosh norm

$$\|x\|_A := \left(\int_0^\infty \|\psi_t(A)x\|^2 \, \frac{dt}{t}\right)^{\frac{1}{2}},$$

where $\psi \in H^\infty(\Omega_\theta)$, $|\psi(z)| \le \frac{|z|^s}{1+|z|^{2s}}$ $(z \in \Omega_\theta)$, for some $s > 0$.

In particular, there is an H^∞ calculus for A, considered as an operator on H_A.

Theorem 2 (J. Galé–P. Miana–Y.)

Put $\Lambda = \log A + 2\pi i$. Then

Additional observations

Remark

In general, $\mathcal{O}_{A,C}$ and $\widehat{W}_{A,B}$ are isomorphisms if one replaces H with H_A, where H_A is the completion of $A(I+A)^{-2}H$ w.r. to the McIntosh norm

$$\|x\|_A := \left(\int_0^\infty \|\psi_t(A)x\|^2 \, \frac{dt}{t}\right)^{\frac{1}{2}},$$

where $\psi \in H^\infty(\Omega_\theta)$, $|\psi(z)| \le \frac{|z|^s}{1+|z|^{2s}}$ $(z \in \Omega_\theta)$, for some $s > 0$.

In particular, there is an H^∞ calculus for A, considered as an operator on H_A.

Theorem 2 (J. Galé–P. Miana–Y.)

Put $\Lambda = \log A + 2\pi i$. Then

$$\Lambda^{-r} H \subset H_A \subset \Lambda^r H, \qquad \forall r > \frac{1}{2}.$$