Realization of symmetric analytic functions of non-commuting variables

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Symmetric polynomials

Every symmetric polynomial in two commuting variables z and w can be written as a polynomial in the variables $z+w$ and zw.

What if z and w do not commute?

The polynomial

$$zwz + wzw$$

in non-commuting variables z, w *cannot* be written as $p(z+w, zw+wz)$ for any nc-polynomial p.

There is no finite basis for the ring of symmetric nc-polynomials over $\mathbb{C}$ (M. Wolf, 1936).

Symmetric analytic functions on the bidisc

Let $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$\pi(z, w) = (z + w, zw).$$

If $\varphi : \mathbb{D}^2 \to \mathbb{C}$ is analytic and symmetric in z and w then there exists an analytic function $\Phi : \pi(\mathbb{D}^2) \to \mathbb{C}$ such that $\varphi = \Phi \circ \pi$:

$$\begin{tikzcd} \mathbb{D}^2 \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & \pi(\mathbb{D}^2) \arrow[dl, "\Phi"] \\ & \mathbb{C} & \end{tikzcd}$$

Are there analogous statements for symmetric functions of non-commuting variables? What is the analogue of $\pi(\mathbb{D}^2)$?

The biball B^2

is the non-commutative analogue of the bidisc:

$$B^2 \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} B_n \times B_n,$$

where B_n denotes the open unit ball of the space $\mathcal{M}_n$ of $n \times n$ complex matrices.

It is an nc-domain:

1) it is open in $\bigcup_{n=1}^{\infty} \mathcal{M}_n^2$;
2) if $x = (x^1, x^2) \in B^2$ and $y = (y^1, y^2) \in B^2$ then

$$x \oplus y \stackrel{\text{def}}{=} (x^1 \oplus y^1, x^2 \oplus y^2) \in B^2;$$

3) if $x \in B^2 \cap \mathcal{M}_n^2$ and u is an $n \times n$ unitary then
$$(u^* x^1 u, u^* x^2 u) \in B^2.$$

Symmetric nc-functions on B^2

An *nc-function* on B^2 is a map

$$\varphi : B^2 \to \mathcal{M}^1 \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \mathcal{M}_n$$

such that, for $n \geq 1$,
1) $\varphi(B^2 \cap \mathcal{M}_n^2) \subset \mathcal{M}_n$,
2) φ respects direct sums, and
3) φ respects similarities.

3) means: if $x \in B^2 \cap \mathcal{M}_n^2$ and s is an invertible $n \times n$ matrix such that $s^{-1}xs \in B^2$ then $\varphi(s^{-1}xs) = s^{-1}\varphi(x)s$.

The function φ is *symmetric* if, for every $x = (x^1, x^2) \in B^2$,

$$\varphi(x^1, x^2) = \varphi(x^2, x^1).$$

Bounded symmetric nc-functions on B^2

There is an nc-domain Ω in the space

$$\mathcal{M}^\infty \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \mathcal{M}_n^\infty$$

and an nc-map $\pi : B^2 \to \Omega$ (given by a simple rational expression) such that every bounded symmetric nc-function φ on B^2 factors through π, and conversely.

$$\begin{array}{ccccc} B^2 & & \xrightarrow{\pi} & & \Omega \\ & \varphi \searrow & & \swarrow \Phi & \\ & & \bigcup_n \mathcal{M}_n & & \end{array}$$

Φ can be expressed by means of a non-commutative version of the familiar linear fractional realization formula for functions in the Schur class.

Theorem

There exists an nc-domain Ω in $\mathcal{M}^\infty$ such that the map

$$\pi : B^2 \to \mathcal{M}^\infty : x \mapsto (u, v^2, vuv, vu^2v, \dots),$$

where

$$u = \frac{x^1 + x^2}{2}, \qquad v = \frac{x^1 - x^2}{2},$$

has the following three properties.

1) π is an analytic nc-map from B^2 to Ω;

2) for every bounded symmetric nc-function φ on the biball there exists an analytic nc-function Φ on Ω such that $\varphi = \Phi \circ \pi$;

3) for every $g \in \Omega$ and every contraction T the operator $g(T)$ exists.

A realization formula

For any symmetric analytic nc-function φ on the biball there exist a unitary operator U on ℓ^2 and a contractive operator

$$\begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus \ell^2 \to \mathbb{C} \oplus \ell^2$$

such that the function Φ on Ω defined, for $n \geq 1$ and $g \in \Omega \cap \mathcal{M}_n^\infty$, by

$$\Phi(g) = a\mathbf{1}_n + (\mathbf{1}_n \otimes B)g(U)\left(\mathbf{1} - (\mathbf{1}_n \otimes D)g(U)\right)^{-1}(\mathbf{1}_n \otimes C)$$

is an nc-function satisfying $\varphi = \Phi \circ \pi$.

Another formulation

Identify a sequence $(g^j) \in \mathcal{M}^\infty$ with the formal power series $\sum_{j\geq 0} g^j z^j$. Then

$$\pi(x)(z) = \frac{x^1 + x^2}{2} + \frac{x^1 - x^2}{2} z \left(\mathbf{1} - \frac{x^1 + x^2}{2} z\right)^{-1} \frac{x^1 - x^2}{2}.$$

For $x \in B^2 \cap \mathcal{M}_n^2$ and a unitary operator U on ℓ^2,

$$\begin{aligned}\pi(x)(U) &= u \otimes \mathbf{1} + v^2 \otimes U + vuv \otimes U^2 + \dots \\ &= \frac{x^1 + x^2}{2} \otimes \mathbf{1}_{\ell^2} + \left(\frac{x^1 - x^2}{2} \otimes U\right) \times \\ &\qquad \left(\mathbf{1}_{\mathbb{C}^n \otimes \ell^2} - \frac{x^1 + x^2}{2} \otimes U\right)^{-1} \left(\frac{x^1 - x^2}{2} \otimes \mathbf{1}_{\ell^2}\right),\end{aligned}$$

an operator on $\mathbb{C}^n \otimes \ell^2$.

Realization again

Let φ be a symmetric nc-function on B^2 bounded by 1 in norm. There exist a unitary U on ℓ^2 and a contraction

$$\begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus \ell^2 \to \mathbb{C} \oplus \ell^2$$

such that, for $x \in B^2 \cap \mathcal{M}_n^2$,

$$\begin{aligned}\varphi(x) = a\mathbf{1}_n + (\mathbf{1}_n \otimes B)\pi(x)(U) \times \\ (\mathbf{1} - (\mathbf{1}_n \otimes D)\pi(x)(U))^{-1} (\mathbf{1}_n \otimes C)\end{aligned}$$

where

$$\begin{aligned}\pi(x)(U) = \frac{x^1 + x^2}{2} \otimes \mathbf{1}_{\ell^2} + \left(\frac{x^1 - x^2}{2} \otimes U\right) \times \\ \left(\mathbf{1}_{\mathbb{C}^n \otimes \ell^2} - \frac{x^1 + x^2}{2} \otimes U\right)^{-1} \left(\frac{x^1 - x^2}{2} \otimes \mathbf{1}_{\ell^2}\right).\end{aligned}$$